ME 233 Advanced Control II

Lecture 9
Review of some topics for infinite-horizon control and estimation

(Not in the ME233 Class Notes)

Outline

- Controllability
- Observability
- Stabilizability
- Detectability
- Transmission Zeros

Controllability

Let $A \in \mathcal{R}^{n \times n}, \ B \in \mathcal{R}^{n \times n_u}$

Theorem: The following are equivalent:

1. *(A,B)* is controllable
2. $\text{rank}[B \ AB \ \cdots \ A^{n-1}B] = n$
3. There does not exist T such that

$$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad T^{-1}B = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}$$

4. $\text{rank}[A - \lambda I \ \ B] = n, \quad \forall \lambda \in \mathcal{C}$
5. The eigenvalues of $A+BK$ can be arbitrarily assigned via choice of K

Observability

Let $A \in \mathcal{R}^{n \times n}, \ C \in \mathcal{R}^{n_y \times n}$

Theorem: The following are equivalent:

1. *(C,A)* is observable

2. $$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

3. There does not exist *T* such that

$$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

$$CT = \begin{bmatrix} \bar{C} & 0 \end{bmatrix}$$

Observability

Let $A \in \mathcal{R}^{n \times n},\ C \in \mathcal{R}^{n_y \times n}$

Theorem (cont'd):

4. $\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathcal{C}$

5. The eigenvalues of $A+LC$ can be arbitrarily assigned via choice of L

Stabilizability (discrete-time)

Let $A \in \mathcal{R}^{n \times n}, \; B \in \mathcal{R}^{n \times n_u}$

Theorem: The following are equivalent:

1. *(A,B)* is stabilizable
2. There does not exist T such that

   $$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad T^{-1}B = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}$$

   where $\bar{A}_{22}$ is not Schur
3. $\text{rank}[A - \lambda I \;\; B] = n$ whenever $|\lambda| \geq 1$
4. There exists K such that $A+BK$ is Schur

Detectability (discrete-time)

Let $A \in \mathcal{R}^{n\times n},\ C \in \mathcal{R}^{n_y\times n}$

Theorem: The following are equivalent:

1. *(C,A)* is detectable
2. There does not exist T such that

$$T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \qquad CT = \begin{bmatrix} \bar{C} & 0 \end{bmatrix}$$

where $\bar{A}_{22}$ is not Schur

3. $\mathrm{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n$ whenever $|\lambda| \geq 1$

4. There exists L such that $A+LC$ is Schur

Normalrank of a MIMO transfer function

- Let $Q(z)$ be a matrix transfer function

- Define

$$\text{normalrank}(Q(z)) := \max_{z \in \mathcal{C}} \Big(\text{rank}(Q(z)) \Big)$$

- Example:

$$\text{rank} \begin{bmatrix} z & 1 \\ z^2 & 1 \\ z & 1 \end{bmatrix} \Bigg|_{z=2} = \text{rank} \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 2 & 1 \end{bmatrix} = 2 \quad \Rightarrow \quad \text{normalrank} \begin{bmatrix} z & 1 \\ z^2 & 1 \\ z & 1 \end{bmatrix} = 2$$

even though $\text{rank} \begin{bmatrix} z & 1 \\ z^2 & 1 \\ z & 1 \end{bmatrix} \Bigg|_{z=1} = \text{rank} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = 1$

Transmission zeros

- Let $G(z)$ be a transfer function with the state-space realization

$$G(z) = C(zI - A)^{-1}B + D$$

- $z_0 \in \mathcal{C}$ is called a <u>transmission zero</u> of this realization if

$$\text{rank}\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} < \text{normalrank}\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$$

- MATLAB command: `zero(sys)`

SISO transmission zeros

- Suppose $G(z)=C(zI-A)^{-1}B+D$ is a <u>SISO</u> transfer function that is not identically zero

- Let $G(z) = \frac{b(z)}{a(z)}$ where $a(z)$ and $b(z)$ are polynomials

- Assume without loss of generality that $a(z) = det(zI - A)$

SISO transmission zeros

- Note that $\det \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$ is a polynomial in z

  $\Rightarrow$ continuous function of z

- Choose z_0 such that $\det(A - z_0 I) \neq 0$

$$\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -(A - z_0 I)^{-1} B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A - z_0 I & 0 \\ C & G(z_0) \end{bmatrix}$$

$$\Rightarrow \det \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} \det \begin{bmatrix} I & -(A - z_0 I)^{-1} B \\ 0 & I \end{bmatrix} = \det \begin{bmatrix} A - z_0 I & 0 \\ C & G(z_0) \end{bmatrix}$$

$$\Rightarrow \det \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = \underline{\det(A - z_0 I)}\,\underline{G(z_0)} = \underline{(-1)^n a(z_0)} \frac{b(z_0)}{\underline{a(z_0)}}$$

$$= (-1)^n b(z_0)$$

SISO transmission zeros

Numerator of *G(z)*

- Whenever $\det(z_0 I - A) \neq 0$,

$$\det \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = (-1)^n b(z_0)$$

- By continuity of the left-hand side

$$\det \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = (-1)^n b(z) \qquad \forall z \in \mathcal{C}$$

- Since *b(z)* is not identically 0, $\operatorname{rank} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$ drops if and only if *b(z)=0*

z is a transmission zero if and only if it is a zero of the transfer function *G(z)*

Transmission zeros

Theorem:

Let $D^T D$ be invertible and define

$$\hat{A} := A - B(D^T D)^{-1} D^T C$$

$$\hat{C} := C - D(D^T D)^{-1} D^T C$$

Then λ is a transmission zero of the state-space realization $G(z) = C(zI - A)^{-1}B + D$ if and only if

$$\operatorname{rank} \begin{bmatrix} \hat{A} - \lambda I \\ \hat{C} \end{bmatrix} < n$$

Unobservable mode of $(\hat{C}, \hat{A})$ at $z = \lambda$