ME 233 Advanced Control II

Lecture 8
Discrete Time
Linear Quadratic Gaussian (LQG)
Optimal Control

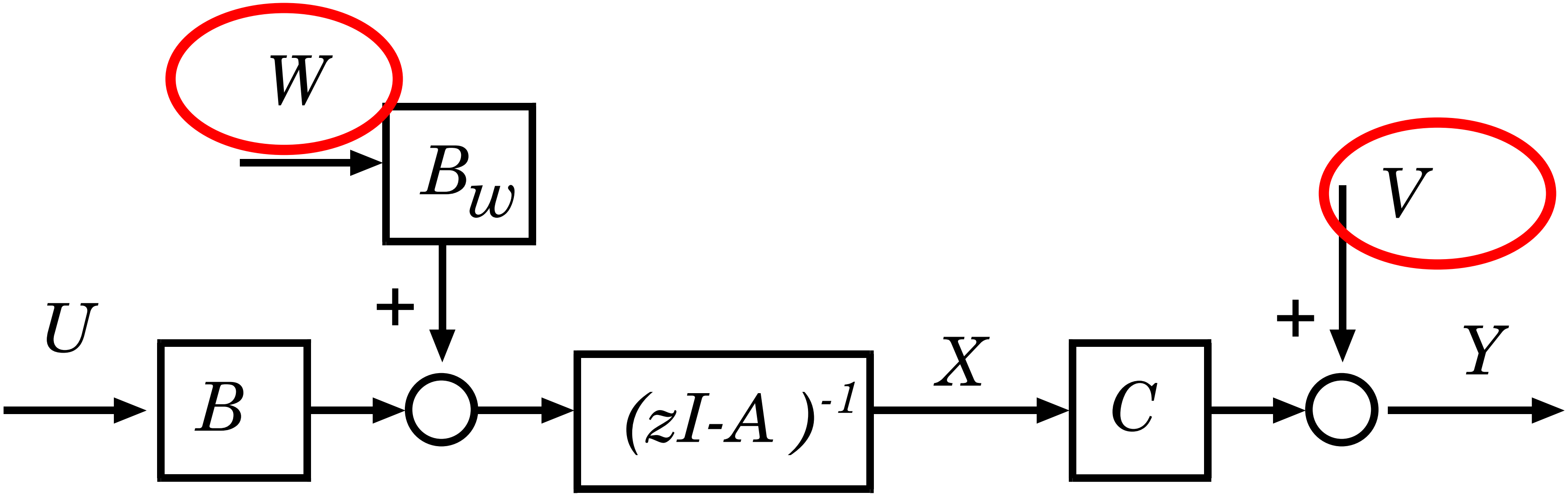
(ME233 Class Notes pp.LQG1-LQG7)

Outline

- Stochastic optimization
- Finite horizon LQG
  - State feedback optimal LQG control
  - Output feedback optimal LQG control

Stochastic Control

Linear system contaminated by noise:



Two random disturbances:

- Input noise $\boldsymbol{w(k)}$ - contaminates the state $\boldsymbol{x(k)}$
- Measurement noise $\boldsymbol{v(k)}$ - contaminates the output $\boldsymbol{y(k)}$

Stochastic state model

$$x(k+1) \;=\; A\,x(k) + B\,u(k) + B_w\,w(k)$$

$$y(k) \;=\; C\,x(k) + v(k)$$

Where:

- $y(k)$ available output
- $u(k)$ **control input**
- $w(k)$ Gaussian, uncorrelated, zero mean, input noise
- $v(k)$ Gaussian, uncorrelated, zero mean, meas. noise
- $x(0)$ Gaussian initial state

Assumptions (same as for KF)

- Initial conditions:

$$E\{x(0)\} = x_o \qquad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o$$

- Noise properties:

$$\left.\begin{aligned}
&E\{w(k)\} = 0\\
&E\{v(k)\} = 0\\
&E\{w(k+l)w^T(k)\} = W(k)\,\delta(l)\\
&E\{v(k+l)v^T(k)\} = V(k)\,\delta(l)\\
&E\{w(k+l)v^T(k)\} = 0
\end{aligned}\right\}$$

Zero-mean Gaussian uncorrelated noises

$$E\{\tilde{x}^o(0)w^T(k)\} = 0 \qquad E\{\tilde{x}^o(0)v^T(k)\} = 0$$

Some notation- control and measurements

The control sequence **<u>from *k* to *N-1*</u>**

$$U_k = \Big(u(k),\, u(k+1),\, \cdots\,,\, u(N-1)\Big)$$

The optimal control sequence **<u>from *k* to *N-1*</u>**

$$U_k^o = \Big(u^o(k),\, u^o(k+1),\, \cdots\,,\, u^o(N-1)\Big)$$

The output measurements **<u>up to *k*</u>**

$$Y_k = \Big(y(0),\; y(1),\; \ldots,\; y(k)\Big)$$

Finite-horizon LQG

For $N > 0$, find the optimal control sequence:

$$U_0^o = \Big(u^o(0),\, u^o(1),\, \cdots,\, u^o(N-1)\Big)$$

Which minimizes the cost functional:

$$J = E\left\{x^T(N)Q_f\, x(N) + \sum_{k=0}^{N-1}\left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\right)\right\}$$

where $u^o(k)$ can only be based on the observations

$$Y_k = \Big(y(0),\; y(1),\; \ldots,\; y(k)\Big)$$

Separation Principle

Main Theorem:

The optimal control is given by:

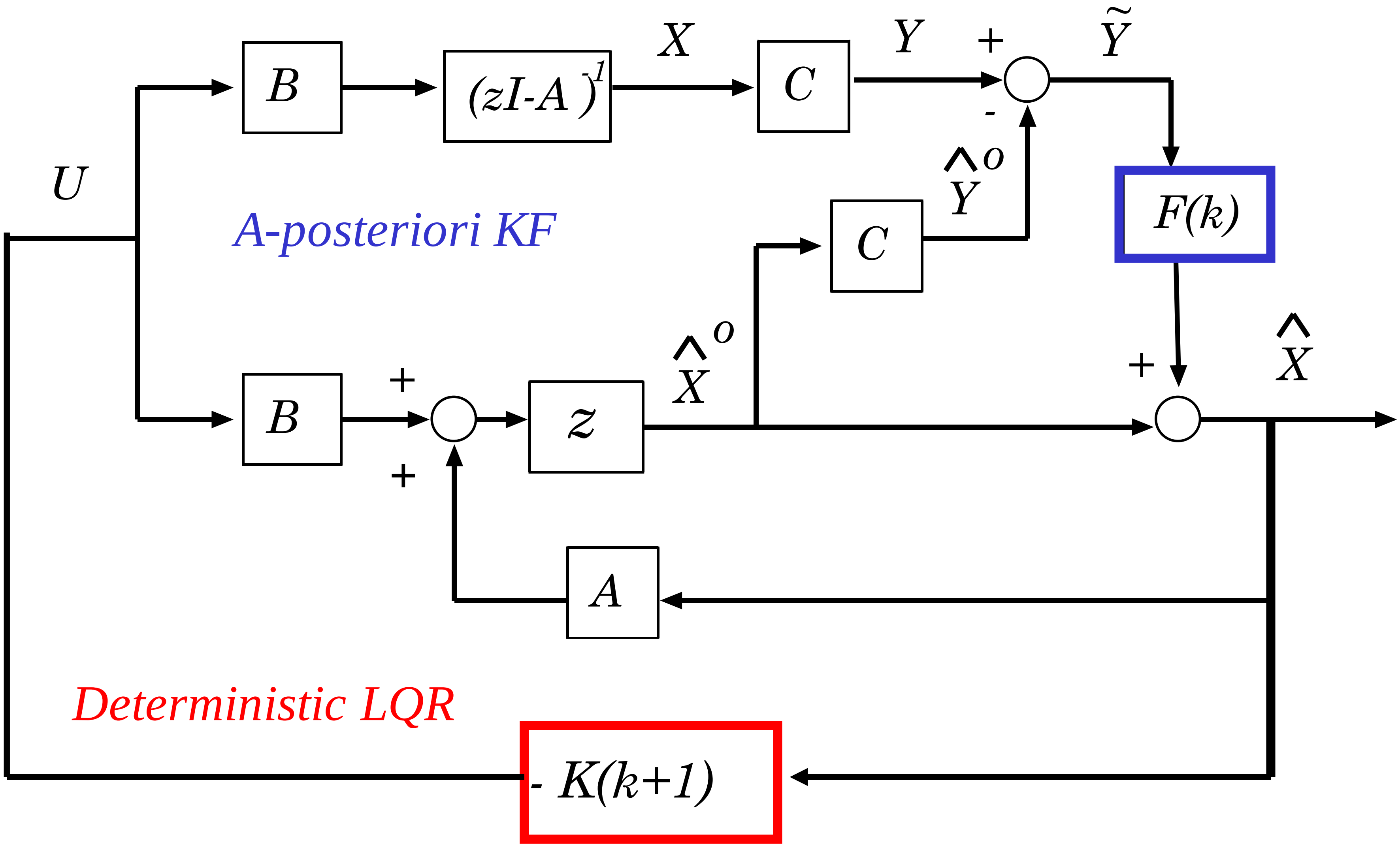
$$\boxed{u^o(k) = -K(k+1)\,\hat{x}(k)}$$

Where:

- The feedback gain $K(k)$ is obtained from the deterministic LQR solution.

- The state estimate $\hat{x}(k)$ is the <u>**a-posteriori**</u> Kalman Filter state estimate.

Separation Principle



Separation Principle Proof

The proof of the separation principle is conducted in two steps:

1. Solve the LQG problem under the assumption that the state vector $x(k)$ is measurable

2. Solve the LQG problem and show that the optimal solution is obtained by replacing $x(k)$ by the a-posteriori state estimate $\widehat{x}(k)$

Finite-horizon state feedback LQG

This problem is similar to the standard deterministic finite-horizon LQR…

$$x(k+1) \;=\; A\,x(k) + B\,u(k) + B_w\,w(k)$$

…except that there is an additional input noise…

…and the control $u(k)$ is only allowed to be a function of

$$x(0),\ \ldots,\ x(k)$$

Functionality constraint on control

- The control $u(k)$ is only allowed to be a function of $x(0), \ldots, x(k)$

- We write this constraint as

$$u(k) \in \underline{u}(k)$$

- We write the constraints $u(k) \in \underline{u}(k)$ for $k=m,\ldots,N-1$ as

$$U_m \in \underline{U}_m$$

Finite-horizon state feedback LQG

We want to solve using dynamic programming:

$$J^o = \min_{U_0 \in \underline{U}_0} E\left\{ x^T(N) Q_f\, x(N) + \sum_{k=0}^{N-1} \left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right) \right\}$$

Need 2 preliminary results:

1. Functional optimization

2. Stochastic Bellman equation

Functional optimization

Lemma 1:

Let X be a random vector and let $u \in \underline{u}$ denote the constraint that u is a function of X

Also assume that there exists $u^o(x)$ such that

$$\min_u f(x,u) = f(x, u^o(x)), \quad \forall x$$

Then $\min_{u \in \underline{u}} E\{f(X,u)\} = E\left\{\min_u f(X,u)\right\}$

Functional optimization

$$\min_{u \in \underline{u}} E\left\{f(X,u)\right\} = E\left\{\min_{u} f(X,u)\right\}$$

Proof is in 2 parts:

1. $$\min_{u \in \underline{u}} E\left\{f(X,u)\right\} \leq E\left\{\min_{u} f(X,u)\right\}$$

2. $$\min_{u \in \underline{u}} E\left\{f(X,u)\right\} \geq E\left\{\min_{u} f(X,u)\right\}$$

$$\min_{u\in\underline{u}} E\left\{f(X,u)\right\} \leq E\left\{\min_u f(X,u)\right\}$$

u is a function of X

Proof:

Let $u^o(x)$ minimize $f(x,u)$

$$\min_u f(x,u) = f(x,u^o(x)),\ \forall x$$

$$\Rightarrow \min_u f(X,u) = f(X,u^o(X))$$

$$\Rightarrow E\left\{\min_u f(X,u)\right\} = E\{f(X,u^o(X))\}$$

$$\geq \min_{u\in\underline{u}} E\left\{f(X,u)\right\}$$

Because $u^o \in \underline{u}$

$\blacksquare$

$$\min_{u \in \underline{u}} E\{f(X,u)\} \geq E\left\{\min_u f(X,u)\right\}$$

u is a function of X

Proof:

- Let $\bar{u} \in \underline{u}$

$$\Rightarrow \min_u f(x,u) \leq f(x, \bar{u}(x)), \ \forall x$$

$$\Rightarrow \min_u f(X,u) \leq f(X, \bar{u}(X))$$

$$\Rightarrow E\left\{\min_u f(X,u)\right\} \leq E\{f(X, \bar{u}(X))\}$$

This holds, regardless of how $\bar{u} \in \underline{u}$ *was chosen*

- Minimizing the right-hand side over $\bar{u} \in \underline{u}$ completes the proof ■

Definitions

- Terminal cost

$$L_f[x(N)] = x^T(N)Q_f x(N)$$

- Stage cost (transient cost)

$$L[x(k), u(k)] = \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

- Optimal cost to go

$$J_N^o = E\{L_f[x(N)]\}$$

$$J_m^o = \min_{U_m \in \underline{U}_m} E\left\{ L_f[x(N)] + \sum_{\underline{k=m}}^{N-1} L[x(k), u(k)] \right\}$$

$m = 0, \ldots, N-1$

Stochastic Bellman equation

Lemma 2:

If $u(k) \in \underline{u}(k)$ for $k = 0, \ldots, m-1$

Then

$$\boxed{J_m^o = \min_{u(m) \in \underline{u}(m)} \left(E\{L[x(m), u(m)]\} + J_{m+1}^o \right) \qquad m = 0, \ldots, N-1}$$

$$J_m^o = \min_{u(m)\in\underline{u}(m)} \left(E\{L[x(m), u(m)]\} + J_{m+1}^o \right)$$

Proof: (m=N-1 case is trivial, and thus omitted)

$$J_m^o = \min_{U_m \in \underline{U}_m} E\left\{ L_f[x(N)] + \sum_{k=m}^{N-1} L[x(k), u(k)] \right\}$$

$$= \min_{u(m)\in\underline{u}(m)} \min_{U_{m+1}\in\underline{U}_{m+1}} \left(E\{L[x(m), u(m)]\} + E\left\{ L_f[x(N)] + \sum_{k=m+1}^{N-1} L[x(k), u(k)] \right\} \right)$$

$$= \min_{u(m)\in\underline{u}(m)} \left(E\{L[x(m), u(m)]\} + \underbrace{\min_{U_{m+1}\in\underline{U}_{m+1}} E\left\{ L_f[x(N)] + \sum_{k=m+1}^{N-1} L[x(k), u(k)] \right\}}_{J_{m+1}^o} \right) \quad \blacksquare$$

Finite-horizon state feedback LQG

Theorem 1:

a) The optimal control is given by

$$u^o(k) = -K(k+1)\,x(k)$$

$$K(k+1) = \left[B^T P(k+1)B + R\right]^{-1}[B^T P(k+1)A + S^T]$$

$$\begin{aligned} P(k-1) = {} & A^T P(k)A + Q \\ & -[A^T P(k)B + S][B^T P(k)B + R]^{-1}[B^T P(k)A + S^T] \end{aligned}$$

$$P(N) = Q_f$$

Standard deterministic LQR solution!

Finite-horizon <u>state feedback</u> LQG

Theorem 1:

b) The optimal cost $\boldsymbol{J^o}$ is given by

$$J^o = x_o^T P(0) x_o + \text{trace}\left[P(0) X_o\right] + b(0)$$

$$x_o = E\{x(0)\} \qquad\qquad X_o = E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}$$

$$b(k) = b(k+1) + \text{trace}\Big[B_w^T P(k+1) B_w W(k)\Big]$$
$$b(N) = 0$$

Finite-horizon state feedback LQG

Theorem 1:

b) The optimal cost is given by

$$J^o = x_o^T P(0) x_o + \text{trace}\left[P(0) X_o\right] + b(0)$$

b(k) *is a dynamic function of the noise intensity*

$b(k)$ *is computed backwards in time with* ***b(N) = 0***

$$b(k) = b(k+1) + \text{trace}\left[B_w^T P(k+1) B_w W(k)\right]$$

This term reflects the detrimental effect of ***w(k)*** *on the cost*

Finite-horizon state feedback LQG

Theorem 1:

b) The optimal cost is given by

$$J^o = \boxed{x_o^T P(0) x_o} + \boxed{\mathrm{trace}\left[P(0) X_o\right]} + \boxed{b(0)}$$

Deterministic LQR cost associated with mean of $x(0)$

Detrimental effect of randomness of $x(0)$ on the cost

Detrimental effect of $w(0),\ldots,w(k)$ on the cost

Finite-horizon state feedback LQG

Proof consists of 2 steps:

1. Prove $J_m^o = E\{x^T(m)P(m)x(m)\} + b(m)$ and $u^o(k) = -K(k+1)x(k)$ using induction on decreasing m, Lemma 1, and the stochastic Bellman equation (Lemma 2)

2. Prove

$$E\{x^T(0)\,P(0)\,x(0)\} = x_0^T P(0) x_0 + \mathrm{trace}[P(0)X_0]$$

$$x_0 = E\{x(0)\}$$
$$X_0 = E\{(x(0) - x_0)(x(0) - x_0)^T\}$$

Proof of Theorem 1: J_m^o and $u^o(m)$

Start with base case: $m=N$

$$J_m^o = E\{L_f[x(N)]\}$$

$$= E\{x^T(N)Q_f x(N)\} + 0$$

$$P(N) \qquad\qquad b(N)$$

$$= E\{x^T(N)P(N)x(N)\} + b(N)$$

Proof of Theorem 1: J_m^o and $u^o(m)$

<u>For *m=0,1,...,N-1*:</u>

(We use induction on decreasing *m*)

By the induction hypothesis,

$$J_{m+1}^o = E\{x^T(m+1)P(m+1)\underbrace{x(m+1)}_{\big(Ax(m)+Bu(m)\big)+B_w w(m)}\} + b(m+1)$$

$$J_{m+1}^o = E\left\{\big(Ax(m)+Bu(m)\big)^T P(m+1)\big(Ax(m)+Bu(m)\big)\right\} \leftarrow \text{Term 1}$$

$$+2E\left\{\big(Ax(m)+Bu(m)\big)^T P(m+1)B_w w(m)\right\} \longleftarrow \text{Term 2}$$

$$+E\left\{w^T(m)B_w^T P(m+1)B_w w(m)\right\} + b(m+1) \longleftarrow \text{Term 3}$$

Proof of Theorem 1: J_m^o and $u^o(m)$

$$2E\left\{\left(Ax(m)+Bu(m)\right)^T P(m+1)B_w w(m)\right\}$$ ← Term 2

Since $x(m)$ and $u(m)$ only depend on quantities that are independent from $w(m)$

$Ax(m) + Bu(m)$ is independent from $w(m)$

$$2E\left\{\left(Ax(m)+Bu(m)\right)^T P(m+1)B_w w(m)\right\}$$

$$= 2E\left\{\left(Ax(m)+Bu(m)\right)^T\right\} P(m+1)B_w \underbrace{E\left\{w(m)\right\}}_{0}$$

$$= 0$$

Proof of Theorem 1: J_m^o and $u^o(m)$

$$E\left\{w^T(m)B_w^T P(m+1)B_w w(m)\right\} + b(m+1)$$ ← Term 3

$$= \mathsf{trace}\left[E\left\{B_w^T P(m+1)B_w w(m)w^T(m)\right\}\right]+b(m+1)$$

$$= \mathsf{trace}\left[B_w^T P(m+1)B_w \underbrace{E\left\{w(m)w^T(m)\right\}}_{W(m)}\right]+b(m+1)$$

$$= b(m)$$

Proof of Theorem 1: J_m^o and $u^o(m)$

Therefore

$$J_{m+1}^o = E\left\{\big(Ax(m) + Bu(m)\big)^T P(m+1)\big(Ax(m) + Bu(m)\big)\right\} + b(m)$$

$$= E\left\{\begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(m) \\ u(m) \end{bmatrix}\right\} + b(m)$$

Proof of Theorem 1: J_m^o and $u^o(m)$

$$J_{m+1}^o = E\left\{ \begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} \right\} + b(m)$$

Now use stochastic Bellman equation

$$J_m^o = \min_{u(k)\in \underline{u}(k)} \underbrace{\left[E\{L[x(m), u(m)]\} + J_{m+1}^o \right]}$$

$$E\left\{ \begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} + \begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} \right\} + b(m)$$

$$= E\left\{ \begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \left(\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} \right\} + b(m)$$

Proof of Theorem 1: J_m^o and $u^o(m)$

$$J_m^o = \min_{u(m)\in\underline{u}(m)} \left[b(m) \right.$$

$$\left. + E\left\{ \begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \left(\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} \right\} \right]$$

- $b(m)$ does not depend on $u(m)$

$$= b(m) + \min_{u(m)\in\underline{u}(m)} E\left\{ \begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \left(\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} \right\}$$

- Use Lemma 1 to exchange min and E

$$= b(m) + E\left\{ \min_{u(m)} \left(\begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \left(\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} \right) \right\}$$

Proof of Theorem 1: J_m^o and $u^o(m)$

$$J_m^o = b(m)$$

$$+E\left\{ \underbrace{\min_{u(m)} \left(\begin{bmatrix} x(m) \\ u(m) \end{bmatrix}^T \left(\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} P(m+1) \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} x(m) \\ u(m) \end{bmatrix} \right)} \right\}$$

This is the same optimization we solved for deterministic LQR!

Optimal value: $x^T(m)P(m)x(m)$

$$u^o(m) = -[B^T P(m+1)B + R]^{-1}[B^T P(m+1)A + S^T]x(m)$$

$$\Longrightarrow J_m^o = b(m) + E\{x^T(m)P(m)x(m)\} \qquad \blacksquare$$

Finite-horizon state feedback LQG

Proof consists of 2 steps:

1. Prove $J_m^o = E\{x^T(m)P(m)x(m)\} + b(m)$ and $u^o(k) = -K(k+1)x(k)$ using induction on decreasing m, Lemma 1, and the stochastic Bellman equation (Lemma 2)

2. Prove

$$E\{x^T(0)\,P(0)\,x(0)\} = x_0^T P(0) x_0 + \text{trace}[P(0)X_0]$$

$$x_0 = E\{x(0)\}$$
$$X_0 = E\{(x(0) - x_0)(x(0) - x_0)^T\}$$

$$E\{x^T(0)\,P(0)\,x(0)\} = x_0^T P(0) x_0 + \mathrm{trace}[P(0)X_0]$$

Proof:

$$(x(0) - x_0) + x_0$$

$$E\{x^T(0)\,P(0)\,x(0)\}$$

$$= E\{(x(0) - x_0)^T P(0)(x(0) - x_0)\}$$

$$+ x_0^T P(0) x_0 + 2\cancelto{0}{E\{(x(0) - x_0)^T\}} P(0) x_0$$

$$= x_0^T P(0) x_0 + \mathrm{trace}\Big[E\{P(0)(x(0) - x_0)(x(0) - x_0)^T\}\Big]$$

$$E\{x^T(0)\,P(0)\,x(0)\} = x_0^T P(0) x_0 + \mathrm{trace}[P(0) X_0]$$

Proof: (cont'd)

$$E\{x^T(0)\,P(0)\,x(0)\}$$

$$= x_0^T P(0) x_0 + \mathrm{trace}\Big[\underbrace{E\{P(0)(x(0)-x_0)(x(0)-x_0)^T\}}_{P(0)E\{(x(0)-x_0)^T(x(0)-x_0)\}}\Big]$$

$$= P(0) X_0$$

$\blacksquare$

Separation Principle Proof

The proof of the separation principle is conducted in two steps:

1. Solve the LQG problem under the assumption that the state vector $x(k)$ is measurable

2. Solve the LQG problem and show that the optimal solution is obtained by replacing $x(k)$ by the a-posteriori state estimate $\widehat{x}(k)$

Finite-horizon LQG

This problem is similar to the standard deterministic finite-horizon LQR…

$$x(k+1) \; = \; A\,x(k) + B\,u(k) + B_w\,w(k)$$

…except that there is an additional input noise…

…and the control $u(k)$ is only allowed to be a function of

$$\underline{Y_k = \Big(y(0), \; \ldots, \; y(k) \Big)}$$

Functionality constraint on control

- The control $u(k)$ is only allowed to be a function of $y(0),\ldots,y(k)$

- As before, we write this constraint as

$$u(k) \in \underline{u}(k)$$

- As before, we write the constraints $u(k) \in \underline{u}(k)$ for $k=m,\ldots,N\text{-}1$ as

$$U_m \in \underline{U}_m$$

Finite-horizon LQG

We want to solve:

$$J^o = \min_{U_0 \in \underline{U}_0} E\left\{ x^T(N) Q_f\, x(N) + \sum_{k=0}^{N-1} \left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right) \right\}$$

We will relate this to an optimal
<u>state feedback</u> LQG control problem

For simplicity, assume $S = 0$

Reformulation of LQG

- Examine $E\{x^T(k)Q\underbrace{x(k)}\}$

$$\begin{aligned}(x(k)-\hat{x}(k))+\hat{x}(k)\\ =\tilde{x}(k)+\hat{x}(k)\end{aligned}$$

$$E\{x^T(k)Qx(k)\} = E\{\hat{x}^T(k)Q\hat{x}(k)\} + E\{\tilde{x}^T(k)Q\tilde{x}(k)\}$$

$$+2E\{\tilde{x}^T(k)Q\hat{x}(k)\}$$

$$= E\{\widehat{x}^T(k)Q\widehat{x}(k)\} + \text{trace}\left[Q\underbrace{E\{\tilde{x}(k)\tilde{x}^T(k)\}}_{Z(k)}\right]$$

$$+\,2\,\text{trace}\left[QE\{\cancel{\widehat{x}(k)\tilde{x}^T(k)}\}\right]$$

0 (by LS property 1)

Reformulation of LQG

- Therefore,

$$E\{x^T(k)Qx(k)\} = E\{\hat{x}^T(k)Q\hat{x}(k)\} + \mathrm{trace}\,[QZ(k)]$$

- Similarly,

$$E\{x^T(N)Q_f x(N)\} = E\{\hat{x}^T(N)Q_f\hat{x}(N)\} + \mathrm{trace}\left[Q_f Z(N)\right]$$

- Want to apply these identities to LQG

$$J^o = \min_{U_0 \in \underline{U}_0} E\left\{x^T(N)Q_f\, x(N) + \sum_{k=0}^{N-1}\left(x^T(k)Qx(k) + u^T(k)Ru(k)\right)\right\}$$

(Recall that we assumed S = 0)

Reformulation of LQG

$$J^o = \min_{U_0 \in \underline{U}_0} E\left\{ x^T(N) Q_f\, x(N) + \sum_{k=0}^{N-1} \Big(x^T(k) Q x(k) + u^T(k) R u(k) \Big) \right\}$$

$$= \min_{U_0 \in \underline{U}_0} \left(E\left\{ \widehat{x}^T(N) Q_f\, \widehat{x}(N) + \sum_{k=0}^{N-1} \Big(\widehat{x}^T(k) Q \widehat{x}(k) + u^T(k) R u(k) \Big) \right\} + \mathrm{trace}\left[Q_f Z(N) \right] + \sum_{k=0}^{N-1} \mathrm{trace}\left[Q Z(k) \right] \right)$$

$$= \mathrm{trace}\left[Q_f Z(N) \right] + \sum_{k=0}^{N-1} \mathrm{trace}\left[Q Z(k) \right] + \min_{U_0 \in \underline{U}_0} E\left\{ \widehat{x}^T(N) Q_f\, \widehat{x}(N) + \sum_{k=0}^{N-1} \Big(\widehat{x}^T(k) Q \widehat{x}(k) + u^T(k) R u(k) \Big) \right\}$$

Reformulation of LQG

$$J^o = \boxed{\text{trace}\left[Q_f Z(N)\right] + \sum_{k=0}^{N-1} \text{trace}\left[Q Z(k)\right]}$$

Terms minimized by the Kalman filter

$$+ \boxed{\min_{U_0 \in \underline{U}_0} E\left\{ \hat{x}^T(N) Q_f \hat{x}(N) + \sum_{k=0}^{N-1} \left(\hat{x}^T(k) Q \hat{x}(k) + u^T(k) R u(k) \right) \right\}}$$

We will show that this corresponds to a <u>state feedback</u> LQG control problem

Reformulation of LQG

- From the Kalman filter :

$$\hat{x}(k+1) = \hat{x}^o(k+1) + F(k+1)\tilde{y}^o(k+1)$$

$$= A\hat{x}(k) + Bu(k) + F(k+1)\tilde{y}^o(k+1)$$

- Recall that $\tilde{y}^o(k+1)$ is uncorrelated and

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,j) = \Big(CM(k)C^T + V(k)\Big)\delta(j)$$

Reformulation of LQG

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + F(k+1)\tilde{y}^o(k+1)$$

Initial conditions:

$$\hat{x}(0) = x_0 + F(0)\tilde{y}^o(0) \qquad\qquad E\{\hat{x}(0)\} = x_0$$

$$\begin{aligned}
\Lambda_{\hat{x}(0)\hat{x}(0)} &= E\{F(0)\tilde{y}^o(0)\tilde{y}^{oT}(0)F^T(0)\} \\
&= F(0)[CM(0)C^T + V(0)]F^T(0) \\
&= \underbrace{M(0)C^T[CM(0)C^T + V(0)]^{-1}CM(0)}
\end{aligned}$$

Notate this as $\bar{X}_0$

Reformulation of LQG

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + F(k+1)\tilde{y}^o(k+1)$$

Initial conditions:

$$\hat{x}(0) = x_0 + F(0)\tilde{y}^o(0) \qquad\qquad E\{\hat{x}(0)\} = x_0$$

Correlation of $\hat{x}(0)$ with $\tilde{y}^o(k+1)$:

$$\Lambda_{\hat{x}(0)\tilde{y}^o(k+1)} = E\{F(0)\tilde{y}^o(0)\tilde{y}^{oT}(k+1)\}$$

$$= 0, \quad \forall k \geq 0$$

Reformulation of LQG

Want to solve:

$$\min_{U_0 \in \underline{U}_0} E\left\{\hat{x}^T(N) Q_f \hat{x}(N) + \sum_{k=0}^{N-1} \left(\hat{x}^T(k) Q \hat{x}(k) + u^T(k) R u(k)\right)\right\}$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + F(k+1)\tilde{y}^o(k+1)$$

$U_0 \in \underline{U}_0$ → *u(k)* is a function of Y_k

→ *u(k)* is a function of $Y_k, \hat{x}(0), \hat{x}(1), \ldots, \hat{x}(k)$

(because $\hat{x}(0), \hat{x}(1), \ldots, \hat{x}(k)$ *are functions of* Y_k *)*

→ *u(k)* is a function of $\hat{x}(0), \hat{x}(1), \ldots, \hat{x}(k)$

(because $E\{\tilde{y}^o(k+1)|Y_k\} = 0$ *, i.e. knowledge of* Y_k *does not give any "information" about* $\tilde{y}^o(k+1)$ *by LS property 1)*

Reformulation of LQG

Want to solve:

$$\min_{U_0 \in \underline{U}_0} E\left\{ \widehat{x}^T(N) Q_f\, \widehat{x}(N) + \sum_{k=0}^{N-1} \left(\widehat{x}^T(k) Q \widehat{x}(k) + u^T(k) R u(k) \right) \right\}$$

u(k) is a function of $\hat{x}(0), \hat{x}(1), \ldots, \hat{x}(k)$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + F(k+1)\tilde{y}^o(k+1)$$

$$E\{\hat{x}(0)\} = x_0$$

Uncorrelated with $\hat{x}(0)$

$$\Lambda_{\widehat{x}(0)\widehat{x}(0)} = \bar{X}_0$$

This is a <u>state feedback</u> LQG control problem!

⇨ Apply results from first half of lecture

Optimal finite-horizon LQG, *S=0*

Main Theorem:

a) The optimal control is given by

$$u^o(k) = \boxed{-K(k+1)}\,\hat{x}(k)$$

$$K(k+1) = \left[B^T P(k+1) B + R\right]^{-1} B^T P(k+1) A$$

$$\begin{aligned} P(k-1) = {} & A^T P(k) A + Q \\ & - A^T P(k) B [B^T P(k) B + R]^{-1} B^T P(k) A \end{aligned}$$

$$P(N) = Q_f$$

Standard deterministic LQR solution!

Optimal finite-horizon LQG, *S=0*

Main Theorem: $u^o(k) = -K(k+1)\,\boxed{\hat{x}(k)}$

A-posteriori state observer structure:

$$\boxed{\begin{aligned} \hat{x}(k) &= \hat{x}^o(k) + F(k)\,\tilde{y}^o(k) \\ \hat{x}^o(k+1) &= A\,\hat{x}(k) + B\,u(k) \\ \tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k) \end{aligned}}$$

$$\boxed{\begin{aligned} F(k) &= M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1} \\ M(k+1) &= AM(k)A^T + B_w W(k) B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T \end{aligned}}$$

Optimal finite-horizon LQG, *S=0*

Main Theorem:

b) The optimal cost J^o is given by

$$J^o = \text{trace}\left[Q_f Z(N)\right] + \sum_{k=0}^{N-1} \text{trace}\left[QZ(k)\right] + x_0^T P(0) x_0 + \text{trace}[P(0)\bar{X}_0] + b(0)$$

$$x_o = E\{x(0)\}$$

$$\bar{X}_0 = X_0 C^T [C X_0 C^T + V(0)]^{-1} C X_0$$

$$b(k) = b(k+1) + \text{trace}\left[F^T(k+1)P(k+1)F(k+1)\left(CM(k+1)C^T + V(k+1)\right)\right]$$

$$b(N) = 0$$

State space form of LQG controller

$$\left.\begin{aligned}\hat{x}^o(k+1) &= [A - L(k)C]\hat{x}^o(k) + Bu(k) + L(k)y(k) \\ \hat{x}(k) &= [I - F(k)C]\hat{x}^o(k) + F(k)y(k)\end{aligned}\right\} \text{Kalman filter}$$

$$\left.u^o(k) = -K(k+1)\hat{x}(k)\right\} \text{LQR}$$

Eliminating $\hat{x}(k)$ from the expression for *u^o^(k)* yields

$$u^o(k) = -K(k+1)[I - F(k)C]\hat{x}^o(k) - K(k+1)F(k)y(k)$$

Plugging this expression for $u^o(k)$ into the expression for $\hat{x}^o(k+1)$ yields the state space model on the next slide

State space form of LQG controller

$$\hat{x}^o(k+1) = A_c(k)\hat{x}^o(k) + B_c(k)y(k)$$

$$u^o(k) = C_c(k)\hat{x}^o(k) + D_c(k)y(k)$$

where

$$A_c(k) = A - L(k)C - BK(k+1) + BK(k+1)F(k)C$$

$$B_c(k) = L(k) - BK(k+1)F(k)$$

$$C_c(k) = -K(k+1) + K(k+1)F(k)C$$

$$D_c(k) = -K(k+1)F(k)$$

K(*k+1*) is the standard deterministic LQR gain

F(*k*) and *L*(*k*) are the standard Kalman filter gains