ME 233 Advanced Control II

Lecture 7
Discrete Time Kalman Filter

(ME233 Class Notes pp.KF1-KF6)

Course Outline

- Unit 0: Probability

Finished

- Unit 1: State-space control, estimation
- Unit 2: Input/output control
- Unit 3: Adaptive control

Wiener Filtering

Norbert Wiener:

- Well-known as the founder of cybernetics, a field he developed in the 1970s that emphasized the modeling of humans as communication and control systems.

- In 1942 he did significant work in the use of time series for military applications; an example of which would be the prediction of the location of enemy planes at the next time step.

- His work in filtering, prediction and smoothing came about in 1949. Wiener filtering is solved for Gaussian time series and under certain assumptions, stationary time series.

Rudy Kalman:

- First major contribution was the introduction of the self-tuning regulator in adaptive control.
- Between 1959 and 1964 he wrote a series of seminal papers:
  - First, the new approach to the filtering problem, known today as Kalman Filtering
  - In the meantime, the all pervasive concept of controllability and its dual, the concept of observability, were formulated.
- By combining the filtering and the control ideas, the first systematic theory for control synthesis, known today as the Linear-Quadratic-Gaussian or LQG theory, resulted.

Deterministic - state feedback

State variable feedback:

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$

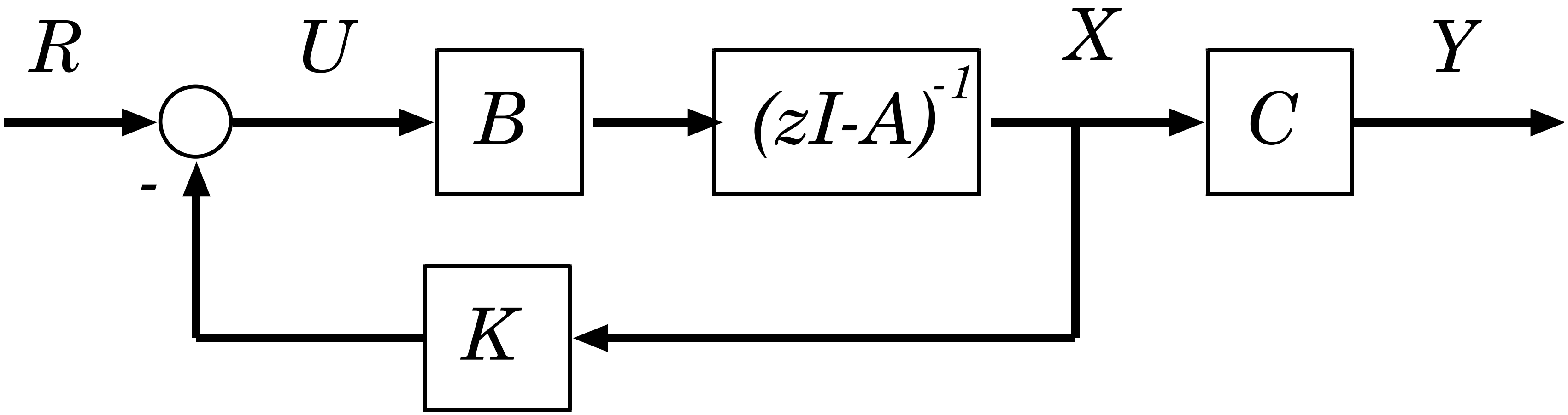
$$u(k) \;=\; -K\,x(k) + r(k)$$

With fictitious reference input $\boldsymbol{r(k)}$

$$r(k) \;=\; r_o \quad = 0$$

Deterministic - state feedback

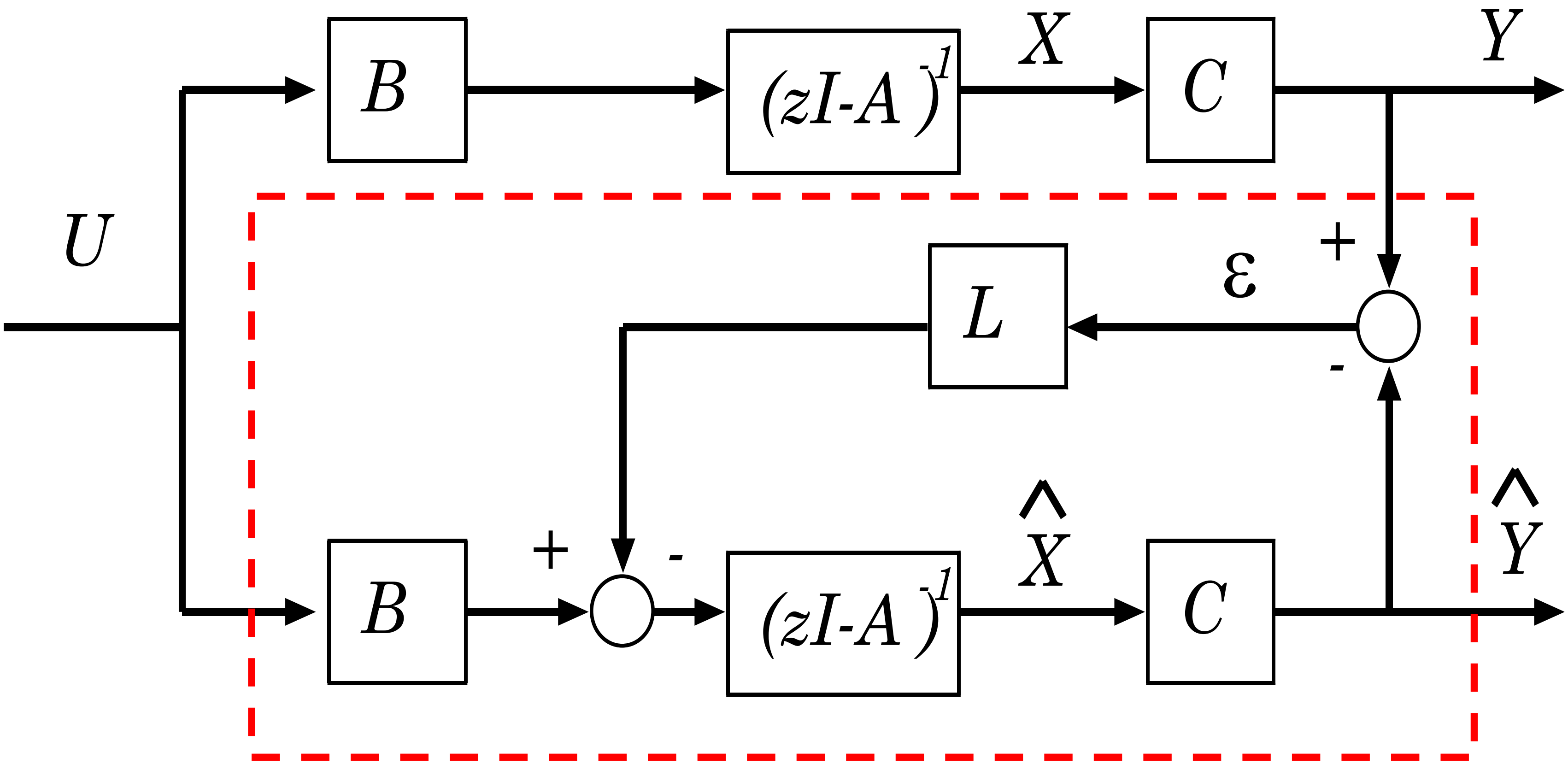
- ME 232 Approach: State Variable Feedback



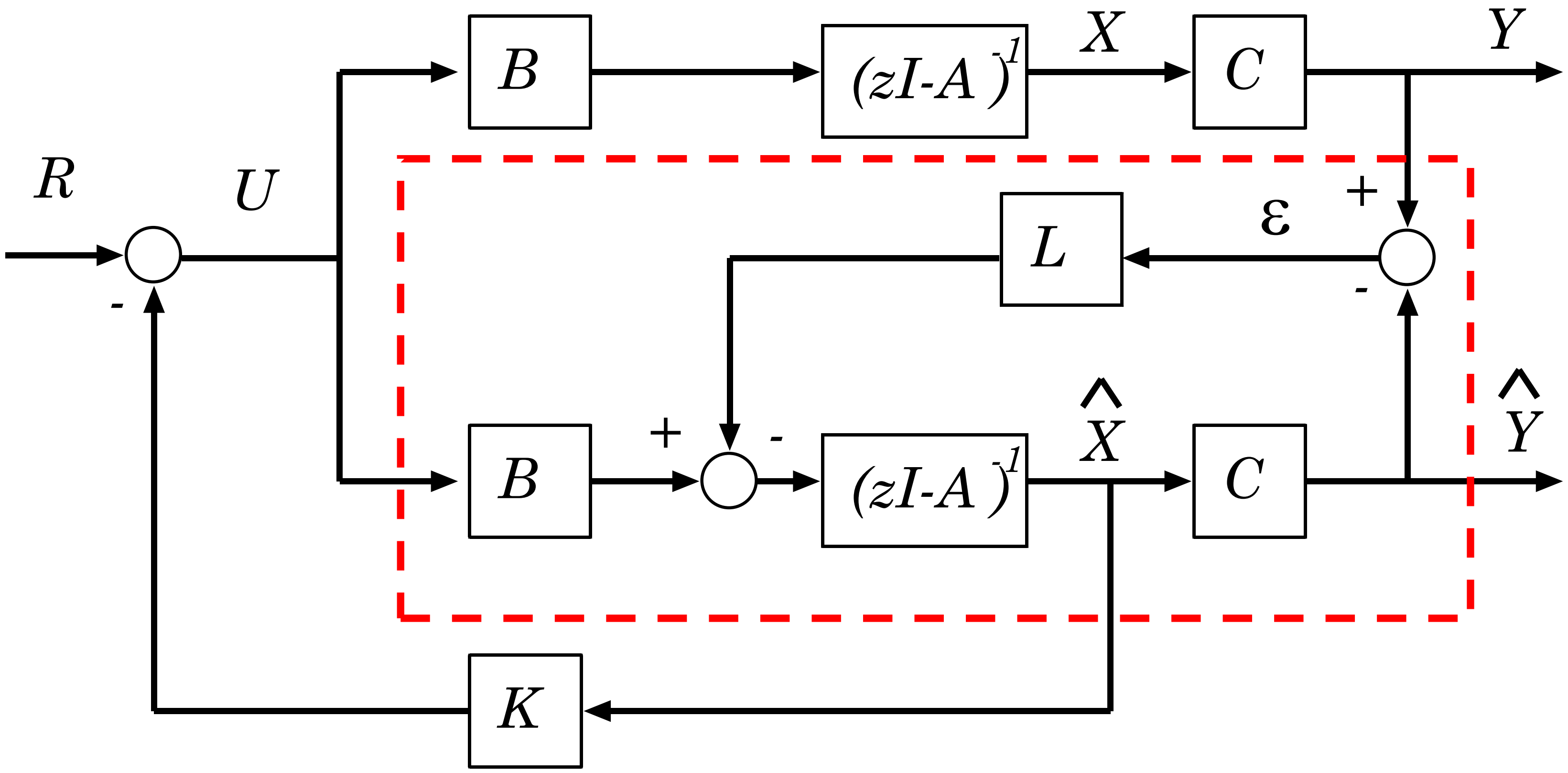
- What happens if the state is not directly measurable – only the output $y(k)$?

Deterministic– state estimation

- ME 232 Approach: State observer

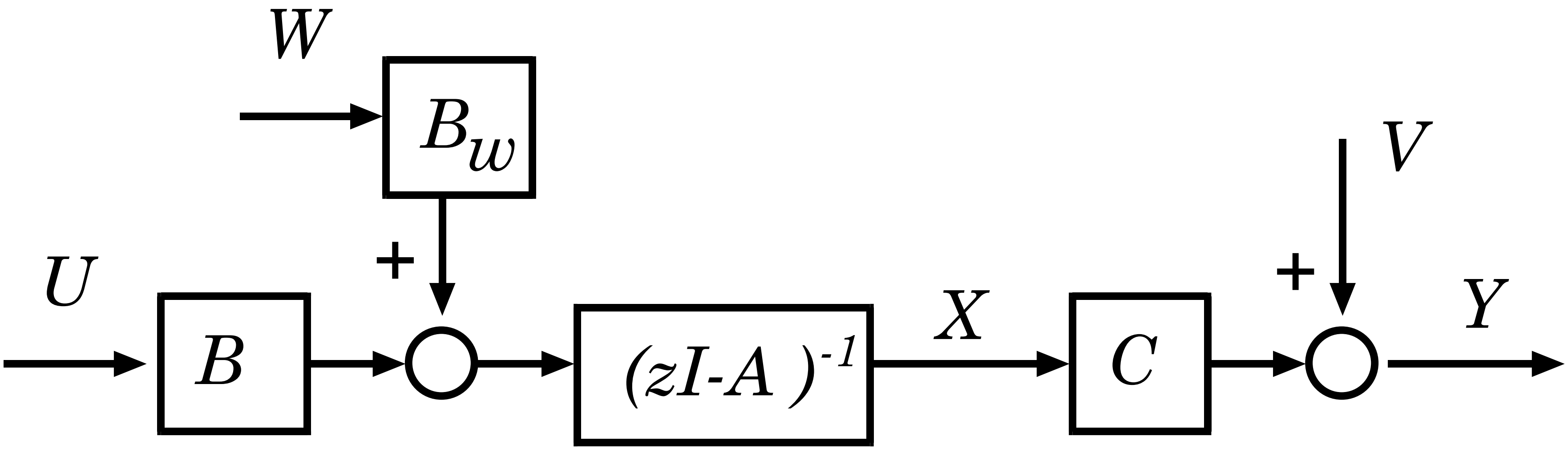


Deterministic– state observer feedback



Stochastic State Estimation

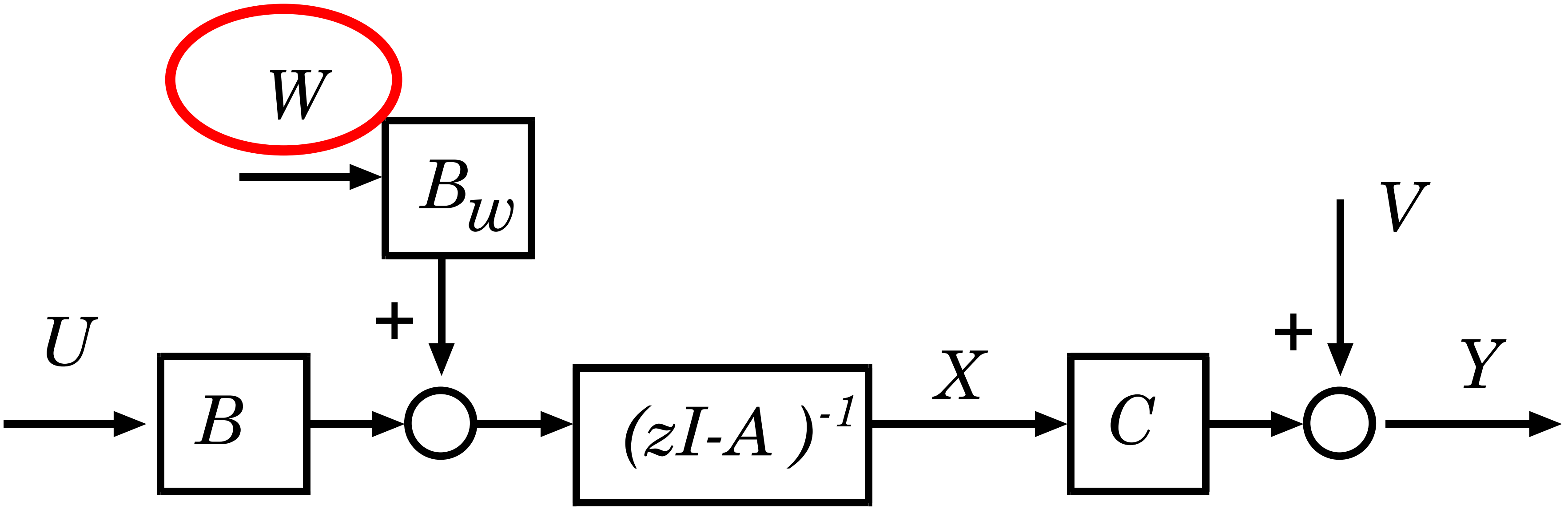
System is now contaminated by noise



Two random disturbances

Stochastic State Estimation

System is now contaminated by noise

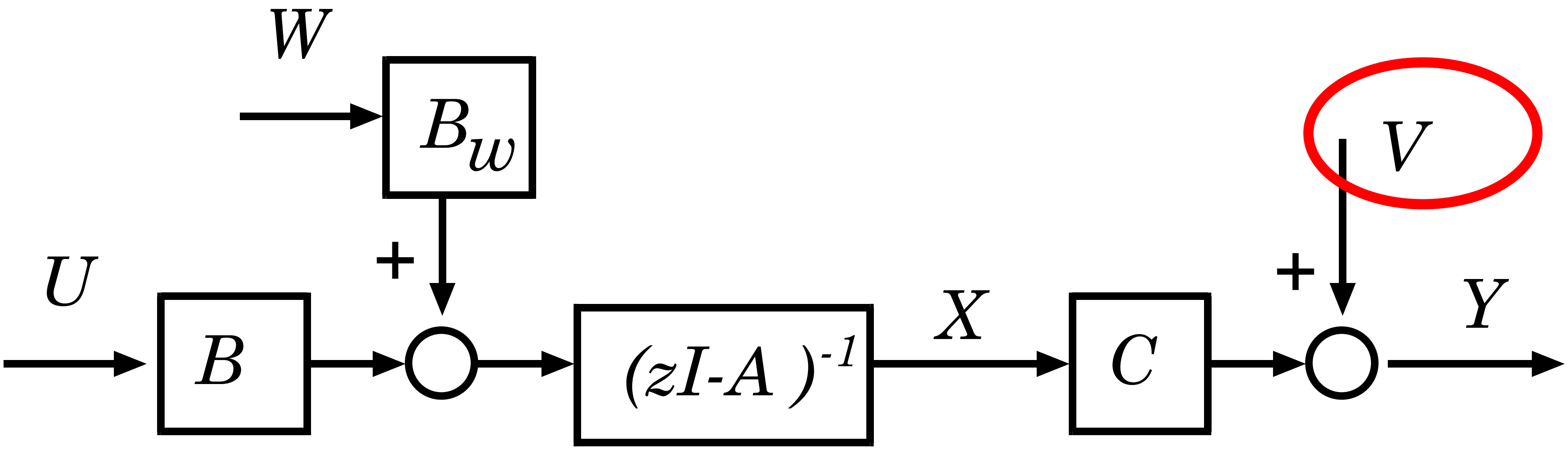


- Input noise $\boldsymbol{w(k)}$ - contaminates the state

$\Rightarrow$ $\boldsymbol{x(k)}$ is now a random sequence

Stochastic State Estimation

System is now contaminated by noise



- Measurement noise $\boldsymbol{v(k)}$ - contaminates the output $\boldsymbol{y(k)}$

Stochastic state model

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\
y(k) &= C\,x(k) + v(k)
\end{aligned}
$$

Where:

- $u(k)$ **known control input**
- $w(k)$ input noise
- $v(k)$ measurement noise

Initial Conditions

- $x(0)$ is Gaussian with **known** marginal mean and covariance:

$$E\{x(0)\} = x_o$$

$$\Lambda_{xx}(0,0) = X_o$$

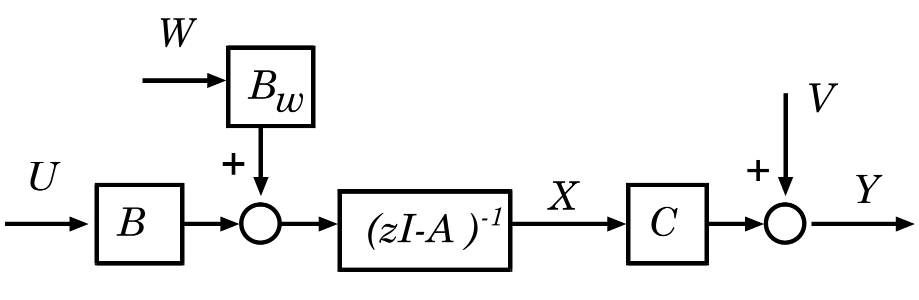
Noises

$w(k)$ and $v(k)$ are:

- **<u>Gaussian</u> zero mean uncorrelated noises**

Not necessarily stationary

- independent from each other and from $x(0)$



Noises

$$E\{w(k)\} = 0$$

$$E\{v(k)\} = 0$$

$$\Lambda_{ww}(k,l) = E\{w(k+l)w^T(k)\} = W(k)\,\delta(l)$$

$$\Lambda_{vv}(k,l) = E\{v(k+l)v^T(k)\} = V(k)\,\delta(l)$$

$$\Lambda_{wv}(k,l) = 0$$

$$E\{(x(0) - x_o)w^T(k)\} = 0$$

$$E\{(x(0) - x_o)v^T(k)\} = 0$$

Output Measurements

$y(k)$ is the measured output, which is considered as an outcome at instant $\boldsymbol{k}$ of the random sequence $\{y(k)\}$ $k = 0,\ 1 \cdots$

- set of available measurements at the instant j

$$Y_j = \{y(0),\ y(1),\ \cdots\ ,\ y(j)\}$$

Notation so far …

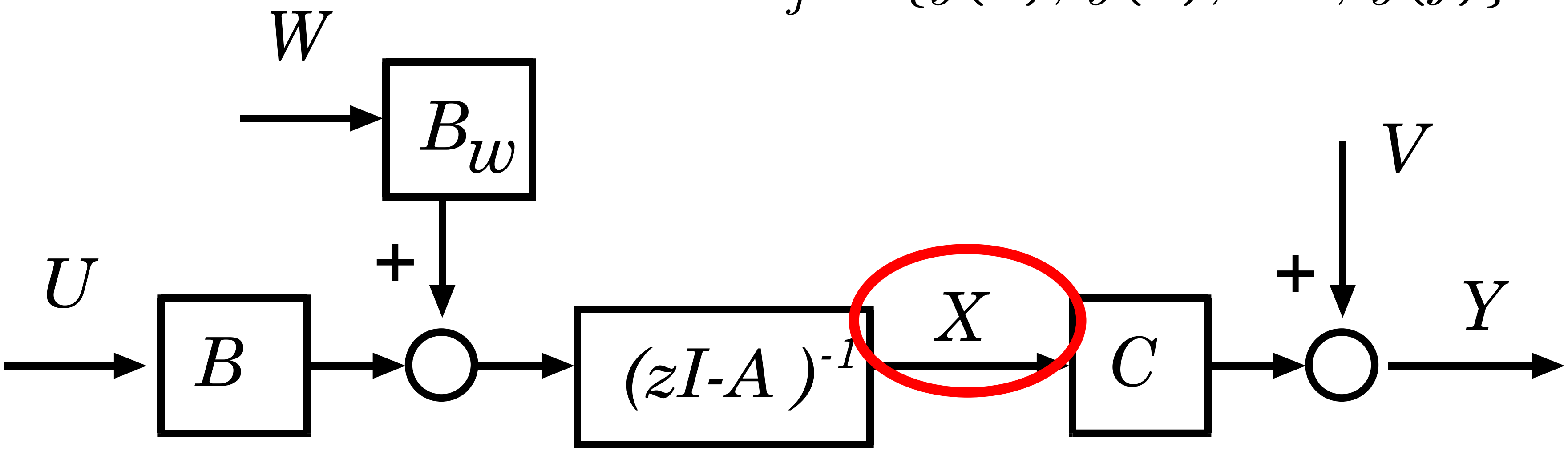
- Initial state marginal mean: x_O
- Initial state marginal covariance: X_O
- Input noise covariance : $W(k)$
- Measurement noise covariance: $V(k)$
- Set of j+1 output measurements: Y_j

  $\{y(0),\, y(1),\, \cdots\, ,\, y(j)\}$

Kalman Filter Objective

Obtain the **best state estimate** given available measurements

$$Y_j = \{y(0), y(1), \cdots, y(j)\}$$



Conditional state estimation problem

Conditional state estimation

New notation:

$$\widehat{x}(k|j) = E\{x(k)|Y_j\}$$

Conditional state estimate

given the set of available measurements:

$$Y_j = \{y(0),\, y(1),\, \cdots,\, y(j)\}$$

Conditional state estimation

$$\widehat{x}(k|j) = E\{x(k)|Y_j\}$$

$$Y_j = \{y(0),\, y(1),\, \cdots\, ,\, y(j)\}$$

When:

- $k = j$ this is a <u>filtering problem</u> ***our focus***
- $k > j$ this is a <u>prediction problem</u>
- $k < j$ this a <u>smoothing problem</u>

A-priori state estimate (one step prediction)

New notation:

$$\widehat{x}^o(k) = \widehat{x}(k|k-1)$$

Conditional state estimate

given the set of available measurements: Y_{k-1}

$$\{y(0),\, y(1),\, \cdots,\, y(k-1)\}$$

before y(k)

A-priori state estimation error:

$$\tilde{x}^o(k) = \tilde{x}(k|k-1) = x(k) - \widehat{x}^o(k)$$

A-posteriori state estimate (filtering)

New notation:

$$\widehat{x}(k) = \widehat{x}(k|k)$$

Conditional state estimate

given the set of available measurements: Y_k

$$\{y(0),\, y(1),\, \cdots,\, y(k)\}$$

after y(k)

A-posteriori state estimation error:

$$\tilde{x}(k) = \tilde{x}(k|k) = x(k) - \widehat{x}(k)$$

State Estimate Covariances

A-priori estimation error covariance:

$$\begin{aligned} M(k) &= E\{\tilde{x}^o(k)\tilde{x}^{oT}(k)\} \\ &= E\{\tilde{x}(k|k-1)\tilde{x}^T(k|k-1)\} \end{aligned}$$

A-posteriori estimation error covariance:

$$\begin{aligned} Z(k) &= E\{\tilde{x}(k)\tilde{x}^T(k)\} \\ &= E\{\tilde{x}(k|k)\tilde{x}^T(k|k)\} \end{aligned}$$

Summary of estimate notation

- $\widehat{x}(k|j) = E\{x(k)|Y_j\}$
- A-priori state estimate : $\widehat{x}^o(k) = \widehat{x}(k|k-1)$
- A-posteriori state estimate : $\widehat{x}(k) = \widehat{x}(k|k)$
- A-priori output estimate :

$$\widehat{y}^o(k) = E\{y(k)|Y_{k-1}\}$$

$$Y_j = \{y(0),\, y(1),\, \cdots,\, y(j)\}$$

Summary of estimate error notation

- A-priori state estimation error and covariance :

$$\tilde{x}^o(k) = x(k) - \hat{x}^o(k)$$

$$M(k) = \Lambda_{\tilde{x}^o(k)\tilde{x}^o(k)}$$

- A-posteriori state estimation error and covariance:

$$\tilde{x}(k) = x(k) - \hat{x}(k)$$

$$Z(k) = \Lambda_{\tilde{x}(k)\tilde{x}(k)}$$

- A-priori output estimation error :

$$\tilde{y}^o(k) = y(k) - \hat{y}^o(k)$$

State Estimate Covariances

Notice that:

$$\text{trace}\, Z(k) \quad \leq \quad \text{trace}\, M(k)$$

A-posteriori *A-priori*

$$E\{\|\tilde{x}(k)\|^2\} \leq E\{\|\tilde{x}^o(k)\|^2\}$$

Initial Conditions for a-priori estimate

Notice that:

$$\widehat{x}^{o}(0) = \widehat{x}(0|-1)$$

a-priori state estimate—before measuring ***y(0)***

$$\widehat{x}^{o}(0) = \widehat{x}(0|-1) = \underbrace{E\{x(0)\} = x_{o}}_{\text{initial state marginal estimation}}$$

Initial Conditions for a-priori estimate

Notice that:

$$M(0) \;=\; E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}$$

$$= E\{(x(0) - x_o)(x(0) - x_o)^T\}$$

$$= \; X_o$$

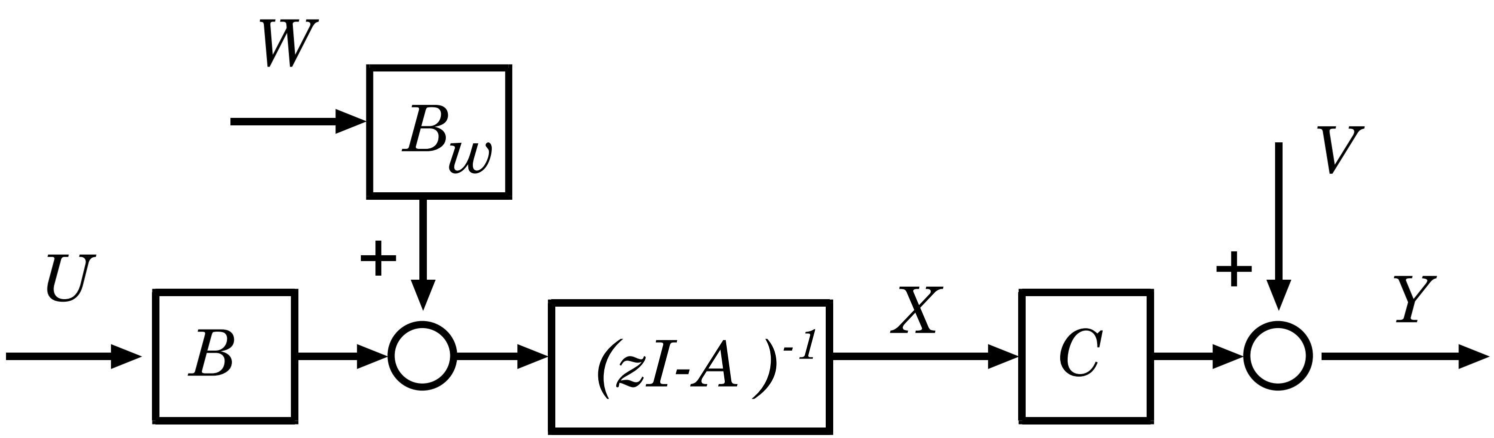
$\hat{x}^o(0)$

initial state
marginal covariance

Kalman Filter Solution

Given:

- I.C.: $\hat{x}^o(0) = x_o$ $\quad M(0) = X_o$
- Noise covariance intensities: $W(k) \quad V(k)$

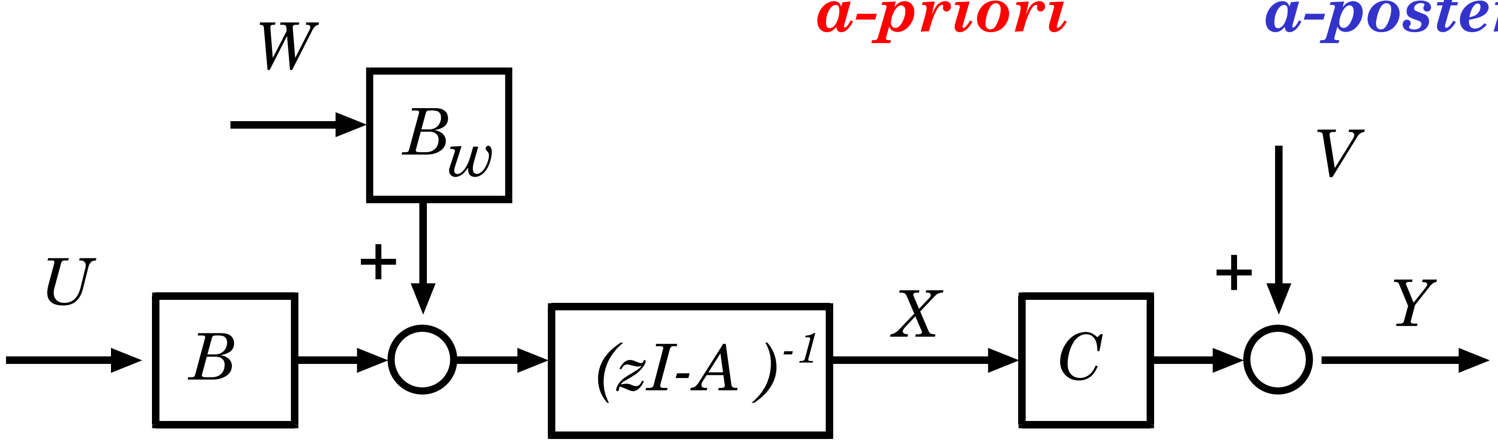


Kalman Filter Solution

Want to recursively find:

- State estimates: $\hat{x}^o(k)$ $\hat{x}(k)$
- Error covariances: $M(k)$ $Z(k)$

a-priori *a-posteriori*



Kalman Filter Solution

Remember:

- Conditional state estimates:

$$\widehat{x}^{o}(k) = \widehat{x}(k|k-1)$$

a-priori (before $y(k)$)

$$\widehat{x}(k) = \widehat{x}(k|k)$$

a-posteriori (after $y(k)$)

Kalman Filter Solution

Remember:

- noises are uncorrelated Gaussian, zero-mean RVSs that are uncorrelated with each other and the initial state:

$$\Lambda_{ww}(k,l) = W(k)\,\delta(l)$$

$$\Lambda_{vv}(k,l) = V(k)\,\delta(l)$$

$$\Lambda_{wv}(k,l) = 0$$

$$\Lambda_{wx}(0,k) = 0$$

$$\Lambda_{vx}(0,k) = 0$$

We will use property 3 of least squares estimation

- Conditional estimator of $\boldsymbol{X}$ given $\boldsymbol{Y}$ and $\boldsymbol{Z}$

$$\hat{X}_{|YZ} = \hat{X}_{|Y} + \left(\tilde{X}_{|Y}\right)_{|(\tilde{Z}_{|Y})}$$

Previous lecture notation:

Notation for Kalman filter:

$$X \longleftrightarrow x(k)$$

$$Y \longleftrightarrow Y_{k-1} = \{y(0), \ldots, y(k-1)\}$$

$$Z \longleftrightarrow y(k)$$

We will use property 3 of least squares estimation

- Conditional estimator of $\boldsymbol{x(k)}$ given $\boldsymbol{Y_{k-1}}$ and $\boldsymbol{y(k)}$

$$\widehat{x}(k) = \widehat{x}^o(k) + \left(\tilde{x}^o(k)\right)_{|(\tilde{y}^o(k))}$$

Previous lecture notation:

Notation for Kalman filter:

$$X \longleftrightarrow x(k)$$

$$Y \longleftrightarrow Y_{k-1} = \{y(0), \ldots, y(k-1)\}$$

$$Z \longleftrightarrow y(k)$$

We will use property 3 of least squares estimation

- Conditional estimation error of $\boldsymbol{X}$ given $\boldsymbol{Y}$ and $\boldsymbol{Z}$

$$\Lambda_{\tilde{X}_{|YZ}\tilde{X}_{|YZ}} = \Lambda_{\tilde{X}_{|Y}\tilde{X}_{|Y}} - \Lambda_{\tilde{X}_{|Y}\tilde{Z}_{|Y}}\Lambda^{-1}_{\tilde{Z}_{|Y}\tilde{Z}_{|Y}}\Lambda_{\tilde{Z}_{|Y}\tilde{X}_{|Y}}$$

Previous lecture notation:		Notation for Kalman filter:
X	$\longleftrightarrow$	$x(k)$
Y	$\longleftrightarrow$	$Y_{k-1} = \{y(0), \ldots, y(k-1)\}$
Z	$\longleftrightarrow$	$y(k)$

We will use property 3 of least squares estimation

- Conditional estimation error of $\boldsymbol{X}$ given $\boldsymbol{Y}$ and $\boldsymbol{Z}$

$$Z(k) = M(k) - \Lambda_{\tilde{x}^o(k)\tilde{y}^o(k)} \Lambda^{-1}_{\tilde{y}^o(k)\tilde{y}^o(k)} \Lambda_{\tilde{y}^o(k)\tilde{x}^o(k)}$$

Previous lecture notation:	Notation for Kalman filter:
X	$x(k)$
Y	$Y_{k-1} = \{y(0), \ldots, y(k-1)\}$
Z	$y(k)$

Kalman Filter Solution: ***k = 0***

- **Before** measurement *y(0)*:

$$\widehat{x}^o(0) = \widehat{x}(0|-1) = E\{x(0)\} = x_o$$

(given)

$$\tilde{x}^o(0) = x(0) - x_o$$

$$M(0) = \Lambda_{\tilde{x}^o(0)\tilde{x}^o(0)}$$

$$= E\{(x(0) - x_o)(x(0) - x_o)^T\}$$

$$= X_o$$ ***(given)***

Kalman Filter Solution: $\boldsymbol{k = 0}$

- A-priori output estimate:

$$\widehat{y}^o(0) = E\{y(0)\} = E\{C\,x(0) + v(0)\}$$

$$= C\,\widehat{x}^o(0) = Cx_o$$

$$(x_o = E\{x(0)\} \neq x(0))$$

A-priori output estimation error (*KF residual*)

$$\widetilde{y}^o(0) = y(0) - C\widehat{x}^o(0) = Cx(0) + v(0) - C\widehat{x}^o(0)$$

$$= C\widetilde{x}^o(0) + v(0)$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

Review of the results so far:

$$\left.\begin{aligned} \hat{x}^o(0) &= x_o \\ \tilde{y}^o(0) &= y(0) - C\,\hat{x}^o(0) \\ M(0) &= X_o \end{aligned}\right\} \textbf{\textit{a-priori}}$$

$$\left.\begin{aligned} \hat{x}(0) &= \\ Z(0) &= \end{aligned}\right\} \textbf{\textit{a-posteriori}}$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

- **After** measurement $y(0)$:

Calculate a-posteriori state estimate using the conditional estimation formula for Gaussians:

$$\hat{x}(0) = \hat{x}^o(0) + \left(\tilde{x}^o(0)\right)_{|(\tilde{y}^o(0))}$$

$$= \hat{x}^o(0) + \Lambda_{\tilde{x}^o(0)\tilde{y}^o(0)} \; \Lambda^{-1}_{\tilde{y}^o(0)\tilde{y}^o(0)} \; \tilde{y}^o(0)$$

(We exploited that $E\{\tilde{x}^o(0)\} = 0, \;\; E\{\tilde{y}^o(0)\} = 0$ *)*

$$\widehat{x}(0) = \widehat{x}^o(0) + \Lambda_{\tilde{x}^o(0)\tilde{y}^o(0)}\ \Lambda^{-1}_{\tilde{y}^o(0)\tilde{y}^o(0)}\ \tilde{y}^o(0)$$

Calculate:

$$\Lambda_{\tilde{x}^o(0)\tilde{y}^o(0)} = E\{\tilde{x}^o(0)\tilde{y}^{oT}(0)\}$$

$$= E\{\tilde{x}^o(0)\,[C\,\tilde{x}^o(0) + v(0)]^T\}$$

$$(E\{\tilde{x}^o(0)v^T(0)\} = 0)$$

$$= \underbrace{E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}}_{M(0)}\,C^T$$

$$= M(0)\,C^T$$

$$\hat{x}(0) = \hat{x}^o(0) + M(0)C^T \; \Lambda^{-1}_{\tilde{y}^o(0)\tilde{y}^o(0)} \; \tilde{y}^o(0)$$

Calculate:

$$\Lambda_{\tilde{y}^o(0)\tilde{y}^o(0)} = E\{\tilde{y}^o(0)\tilde{y}^{oT}(0)\}$$

$$= E\{[C\,\tilde{x}^o(0) + v(0)]\,[C\,\tilde{x}^o(0) + v(0)]^T\}$$

$$(E\{\tilde{x}^o(0)v^T(0)\} = 0)$$

$$= C\,\underbrace{E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\}}_{M(0)}\,C^T + \underbrace{E\{v(0)v^T(0)\}}_{V(0)}$$

$$= C\,M(0)\,C^T + V(0)$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

- **a-posteriori state** estimate:

$$\widehat{x}(0) = \widehat{x}^o(0) + \underbrace{\Lambda_{\tilde{x}^o(0)\tilde{y}^o(0)}}_{M(0)C^T} \underbrace{\Lambda^{-1}_{\tilde{y}^o(0)\tilde{y}^o(0)}}_{[CM(0)C^T + V(0)]^{-1}} \tilde{y}^o(0)$$

$$\boxed{\begin{aligned} \widehat{x}(0) &= \widehat{x}^o(0) + M(0)C^T\left[C\,M(0)C^T + V(0)\right]^{-1}\tilde{y}^o(0) \\ \tilde{y}^o(0) &= y(0) - C\,\widehat{x}^o(0) \qquad \widehat{x}^o(0) = x_o \end{aligned}}$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

Review of the results so far:

$$\left.\begin{aligned} \widehat{x}^o(0) &= x_o \\ \tilde{y}^o(0) &= y(0) - C\,\widehat{x}^o(0) \\ M(0) &= X_o \end{aligned}\right\}\ \textbf{\textit{a-priori}}$$

$$\widehat{x}(0) = \widehat{x}^o(0) + M(0)C^T\left[C\,M(0)C^T + V(0)\right]^{-1}\tilde{y}^o(0)$$

$$Z(0) = E\{\tilde{x}(0)\tilde{x}^T(0)\} = \Lambda_{\tilde{x}(0)\tilde{x}(0)}$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

- **A-posteriori state** estimation error:

$$\tilde{x}(0) \;=\; x(0) - \hat{x}(0)$$

- **A-posteriori state** estimation error covariance:

$$Z(0) \;=\; E\{\tilde{x}(0)\tilde{x}^T(0)\} = \Lambda_{\tilde{x}(0)\tilde{x}(0)}$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

- **a-posteriori state** estimation covariance:

$$Z(0) \;=\; E\{\tilde{x}(0)\tilde{x}^T(0)\} = \Lambda_{\tilde{x}(0)\tilde{x}(0)}$$

- Use least squares result:

$$\Lambda_{\tilde{X}_{|YZ}\tilde{X}_{|YZ}} = \Lambda_{\tilde{X}_{|Y}\tilde{X}_{|Y}} - \Lambda_{\tilde{X}_{|Y}\tilde{Z}_{|Y}}\ \Lambda^{-1}_{\tilde{Z}_{|Y}\tilde{Z}_{|Y}}\ \Lambda_{\tilde{Z}_{|Y}\tilde{X}_{|Y}}$$

$$\Downarrow$$

$$\Lambda_{\tilde{x}(0)\tilde{x}(0)} = \Lambda_{\tilde{x}^o(0)\tilde{x}^o(0)} - \Lambda_{\tilde{x}^o(0)\tilde{y}^o(0)}\ \Lambda^{-1}_{\tilde{y}^o(0)\tilde{y}^o(0)}\ \Lambda_{\tilde{y}^o(0)\tilde{x}^o(0)}$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

- **a-posteriori state** estimation covariance:

$$Z(0) \;=\; E\{\tilde{x}(0)\tilde{x}^T(0)\} = \Lambda_{\tilde{x}(0)\tilde{x}(0)}$$

- Use least squares result:

$$\Lambda_{\tilde{x}(0)\tilde{x}(0)} = \underbrace{\Lambda_{\tilde{x}^o(0)\tilde{x}^o(0)}}_{M(0)} - \underbrace{\Lambda_{\tilde{x}^o(0)\tilde{y}^o(0)}}_{M(0)C^T} \underbrace{\Lambda^{-1}_{\tilde{y}^o(0)\tilde{y}^o(0)}}_{[CM(0)C^T + V(0)]^{-1}} \Lambda_{\tilde{y}^o(0)\tilde{x}^o(0)}$$

$$\boxed{Z(0) \;=\; M(0) - M(0)C^T\left[CM(0)C^T + V(0)\right]^{-1} CM(0)}$$

Kalman Filter Solution: $\boldsymbol{k = 0}$

Review of the results so far:

$$\widehat{x}^o(0) = x_o$$

$$\tilde{y}^o(0) \;=\; y(0) - C\,\widehat{x}^o(0)$$

$$M(0) \;=\; X_o$$

$$\widehat{x}(0) \;=\; \widehat{x}^o(0) + M(0)C^T\left[C\,M(0)C^T + V(0)\right]^{-1}\tilde{y}^o(0)$$

$$Z(0) \;=\; M(0) - M(0)C^T\left[CM(0)C^T + V(0)\right]^{-1}CM(0)$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

Before measurement $y(1)$:

- Determine a-priori state estimate $\widehat{x}^o(1)$

$$\widehat{x}^o(1) = \widehat{x}(1|0) = E\{x(1)|y(0)\}$$

- Determine a-priori state estimation error covariance

$$M(1) = E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

A-priori state estimate:

$$\widehat{x}^o(1) = \widehat{x}(1|0) = E\{x(1)|y(0)\}$$

- **Use state equation and take conditional expectations:**

$$x(1) \;=\; A\,x(0) + B\,u(0) + B_w\,w(0)$$

$$\widehat{x}(1|0) \;=\; A\,\widehat{x}(0|0) + B\,u(0)$$

$$\boxed{\widehat{x}^o(1) \;=\; A\,\widehat{x}(0) + B\,u(0)}$$

Independent from $\boldsymbol{y(0)}$

Kalman Filter Solution: $\boldsymbol{k = 1}$

A-priori state estimation error:

$$\tilde{x}^o(1) = x(1) - \hat{x}^o(1)$$

- **Use state equation:**

$$\begin{aligned} x(1) &= A\,x(0) + B\,u(0) + B_w\,w(0) \\ \hat{x}^o(1) &= A\,\hat{x}(0) + B\,u(0) \end{aligned}$$

$$\boxed{\tilde{x}^o(1) = A\,\tilde{x}(0) + B_w\,w(0)}$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

A-priori state estimation error covariance:

$$M(1) = E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}$$

- **Use:**

$$\tilde{x}^o(1) \;=\; A\,\tilde{x}(0) + B_w\,w(0) \qquad E\{\tilde{x}(0)w^T(0)\}$$

$$\underbrace{E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}}_{M(1)} = A\,\underbrace{E\{\tilde{x}(0)\tilde{x}^T(0)\}}_{Z(0)}A^T$$

$$+\,B_w\,\underbrace{E\{w(0)w^T(0)\}}_{W(0)}B_w^T$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

A-priori state estimation error covariance:

$$M(1) = E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}$$

$$\boxed{M(1) \;=\; A\,Z(0)A^T + B_w W(0) B_w^T}$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

- **Before** measurement $y(1)$:

$$\hat{y}^o(1) = E\{y(1)|y(0)\}$$

$$= E\{C\,x(1) + v(1)|y(0)\}$$

$$= C\,E\{x(1)|y(0)\}$$

$$= C\hat{x}^o(1)$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

Before measurement *y(1)*:

$$\widehat{y}^o(1) \;=\; C\,\widehat{x}^o(1)$$

A-priori output estimation error $\tilde{y}^o(1)$

$$\tilde{y}^o(1) \;=\; y(1) - \widehat{y}^o(1)$$

$$\tilde{y}^o(1) \;=\; y(1) - C\,\widehat{x}^o(1)$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

Review of the results so far:

$$\left.\begin{aligned}
\widehat{x}^o(1) &= A\,\widehat{x}(0) + B\,u(0) \\
\tilde{y}^o(1) &= y(1) - C\,\widehat{x}^o(1) \\
M(1) &= A\,Z(0)A^T + B_w W(0) B_w^T
\end{aligned}\right\} \textbf{\textit{a-priori}}$$

$$\left.\begin{aligned}
\widehat{x}(1) &= \\
Z(1) &=
\end{aligned}\right\} \textbf{\textit{a-posteriori}}$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

- **After** measurement $y(1)$:

Calculate a-posteriori state estimate using the conditional estimation formula for Gaussians:

$$\hat{x}(1) = \hat{x}^o(1) + \left(\tilde{x}^o(1)\right)_{|(\tilde{y}^o(1))}$$

$$= \hat{x}^o(1) + \Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)} \; \Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)} \; \tilde{y}^o(1)$$

(We exploited that $E\{\tilde{x}^o(1)\} = 0, \;\; E\{\tilde{y}^o(1)\} = 0$ *)*

Kalman Filter Solution: $\boldsymbol{k = 1}$

Before measurement $y(1)$:

$$\widehat{y}^o(1) = C\,\widehat{x}^o(1)$$

A-priori output estimation error $\widetilde{y}^o(1)$

$$\begin{aligned}\widetilde{y}^o(1) &= y(1) - \widehat{y}^o(1) \\ &= Cx(1) + v(1) - C\,\widehat{x}^o(1) \\ &= C\,\widetilde{x}^o(1) + v(1)\end{aligned}$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

IMPORTANT: Property 1 of least squares estimation:

$$\widehat{y}^{o}(1) = E\{y(1)|y(0)\}$$

- **The a-priori output estimation error** $\tilde{y}^{o}(1)$ **is uncorrelated with** $y(0)$

$$\boxed{E\{y(0)\tilde{y}^{oT}(1)\} = 0}$$

$$\hat{x}(1) = \hat{x}^o(1) + \Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)} \Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)} \tilde{y}^o(1)$$

- Calculate $\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}$

$$\begin{aligned}
\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)} &= E\{\tilde{x}^o(1)\tilde{y}^{oT}(1)\} \\
&= E\{\tilde{x}^o(1)\,[C\,\tilde{x}^o(1) + v(1)]^T\}
\end{aligned}$$

$$E\{\tilde{x}^o(1)v^T(1)\} = 0$$

$$\begin{aligned}
&= \underbrace{E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}}_{M(1)}\,C^T \\
&= M(1)\,C^T
\end{aligned}$$

$$\hat{x}(1) = \hat{x}^o(1) + M(1)\,C^T\,\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}\,\tilde{y}^o(1)$$

- Calculate $\Lambda_{\tilde{y}^o(1)\tilde{y}^o(1)}$

$$\begin{aligned}
\Lambda_{\tilde{y}^o(1)\tilde{y}^o(1)} &= E\{\tilde{y}^o(1)\tilde{y}^{oT}(1)\} \\
&= E\{[C\,\tilde{x}^o(1) + v(1)]\,[C\,\tilde{x}^o(1) + v(1)]^T\}
\end{aligned}$$

$$E\{\tilde{x}^o(1)v^T(1)\} = 0$$

$$\begin{aligned}
&= C\,\underbrace{E\{\tilde{x}^o(1)\tilde{x}^{oT}(1)\}}_{M(1)}\,C^T + \underbrace{E\{v(1)v^T(1)\}}_{V(1)} \\
&= C\,M(1)\,C^T + V(1)
\end{aligned}$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

- **a-posteriori state** estimate:

$$\widehat{x}(1) \;=\; \widehat{x}^o(1) + \underbrace{\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}}_{M(1)C^T}\underbrace{\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}}_{[C M(1)C^T + V(1)]^{-1}}\tilde{y}^o(1)$$

$$\boxed{\begin{aligned} \widehat{x}(1) \;&=\; \widehat{x}^o(1) + M(1)C^T\left[C\,M(1)C^T + V(1)\right]^{-1}\tilde{y}^o(1) \\ \tilde{y}^o(1) \;&=\; y(1) - C\,\widehat{x}^o(1) \end{aligned}}$$

Kalman Filter Solution: $k = 1$

Review of the results so far:

$$\tilde{x}^o(1) \;=\; A\,\tilde{x}(0) + B_w\,w(0)$$

$$\tilde{y}^o(1) \;=\; y(1) - C\,\hat{x}^o(1)$$

$$M(1) \;=\; A\,Z(0)A^T + B_w W(0) B_w^T$$

$$\hat{x}(1) \;=\; \hat{x}^o(1) + M(1)C^T\left[C\,M(1)C^T + V(1)\right]^{-1}\tilde{y}^o(1)$$

$$Z(1) \;=$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

- **A-posteriori state** estimation error:

$$\tilde{x}(1) \;=\; x(1) - \hat{x}(1)$$

- **A-posteriori state** estimation error covariance:

$$Z(1) \;=\; E\{\tilde{x}(1)\tilde{x}^T(1)\} = \Lambda_{\tilde{x}(1)\tilde{x}(1)}$$

Kalman Filter Solution: *k = 1*

- **a-posteriori state** estimation covariance:

$$Z(1) \;=\; E\{\tilde{x}(1)\tilde{x}^T(1)\} = \Lambda_{\tilde{x}(1)\tilde{x}(1)}$$

- Use least squares result:

$$\underbrace{\Lambda_{\tilde{x}(1)\tilde{x}(1)}}_{Z(1)} = \underbrace{\Lambda_{\tilde{x}^o(1)\tilde{x}^o(1)}}_{M(1)} - \underbrace{\Lambda_{\tilde{x}^o(1)\tilde{y}^o(1)}}_{M(1)C^T}\underbrace{\Lambda^{-1}_{\tilde{y}^o(1)\tilde{y}^o(1)}}_{[CM(1)C^T+V(1)]^{-1}}\Lambda_{\tilde{y}^o(1)\tilde{x}^o(1)}$$

$$\boxed{Z(1) \;=\; M(1) - M(1)C^T\left[CM(1)C^T + V(1)\right]^{-1} CM(1)}$$

Kalman Filter Solution: $\boldsymbol{k = 1}$

Review:

$$
\begin{aligned}
\widehat{x}^o(1) &= A\,\widehat{x}(0) + B\,u(0) \\
\widetilde{y}^o(1) &= y(1) - C\,\widehat{x}^o(1)
\end{aligned}
$$

$$
M(1) = A\,Z(0)A^T + B_w W(0) B_w^T
$$

$$
\widehat{x}(1) = \widehat{x}^o(1) + M(1)C^T\left[C\,M(1)C^T + V(1)\right]^{-1}\widetilde{y}^o(1)
$$

$$
Z(1) = M(1) - M(1)C^T\left[CM(1)C^T + V(1)\right]^{-1}CM(1)
$$

Equations are entirely recursive!

Kalman Filter Solution

1) Compute a-priori output estimation error residual:

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

2) Compute a-posteriori state estimate:

$$\hat{x}(k) \;=\; \hat{x}^o(k) + M(k)C^T \left[C\,M(k)C^T + V(k)\right]^{-1} \tilde{y}^o(k)$$

3) Update a-priori state estimate:

$$\hat{x}^o(k+1) \;=\; A\,\hat{x}(k) + B\,u(k)$$

Kalman Filter Solution

4) Compute a-posteriori state estimation error covariance:

$$Z(k) \;=\; M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)$$

5) Update a-priori state estimation error covariance:

$$M(k+1) \;=\; A\,Z(k)A^T + B_w W(k) B_w^T$$

Kalman filter implementation

$\widehat{x}^o(0) = x_0$

$M(0) = X_0$

for $k = 0, 1, 2, \ldots$

$$\begin{aligned}
&\text{obtain measurement } y(k) \\
&\tilde{y}^o(k) = y(k) - C\widehat{x}^o(k) \\
&\widehat{x}(k) = \widehat{x}^o(k) + M(k)C^T[CM(k)C^T + V(k)]^{-1}\tilde{y}^o(k) \\
&Z(k) = M(k) - M(k)C^T[CM(k)C^T + V(k)]^{-1}CM(k) \\
\\
&\widehat{x}^o(k+1) = A\widehat{x}(k) + Bu(k) \\
&M(k+1) = AZ(k)A^T + B_wW(k)B_w^T \\
&\text{wait for next measurement}
\end{aligned}$$

end

Kalman Filter Solution V-2

- Kalman filter algorithm can be written in a different manner, which only involves the a-priori state estimate and the a-priori estimation error covariance.

$$\hat{x}(k) \;=\; \hat{x}^o(k) + F(k)\,\tilde{y}^o(k)$$

$$F(k) = M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}$$

$$M(0) \;=\; X_o$$

Kalman Filter Solution V-2

Plug

$$\widehat{x}(k) = \underbrace{\widehat{x}^o(k) + F(k)\,\widetilde{y}^o(k)}$$

Into

$$\widehat{x}^o(k+1) = A\,\widehat{x}(k) + B\,u(k)$$

$$\widehat{x}^o(k+1) = A\,[\widehat{x}^o(k) + F(k)\widetilde{y}^o(k)] + B\,u(k)$$

Results in

$$\widehat{x}^o(k+1) = A\,\widehat{x}^o(k) + B\,u(k) + \overbrace{L(k)}^{A\,F(k)}\,\widetilde{y}^o(k)$$

Kalman Filter Solution V-2

$$\widehat{x}^o(k+1) \;=\; A\,\widehat{x}^o(k) + B\,u(k) + L(k)\,\tilde{y}^o(k)$$

where

$$L(k) \;=\; A\,F(k)$$

$$L(k) \;=\; A\,\underbrace{M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1}}_{F(k)}$$

Kalman Filter Solution V-2

Plugging

$$Z(k) \;=\; \underbrace{M(k) - M(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)}$$

Into

$$M(k+1) \;=\; A\,Z(k)A^T + B_w W(k) B_w^T$$

- **Results in the following discrete Riccati difference equation:**

$$\begin{aligned} M(k+1) \;=\;& AM(k)A^T + B_w W(k) B_w^T \\ & - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T \end{aligned}$$

Kalman Filter Solution V-2

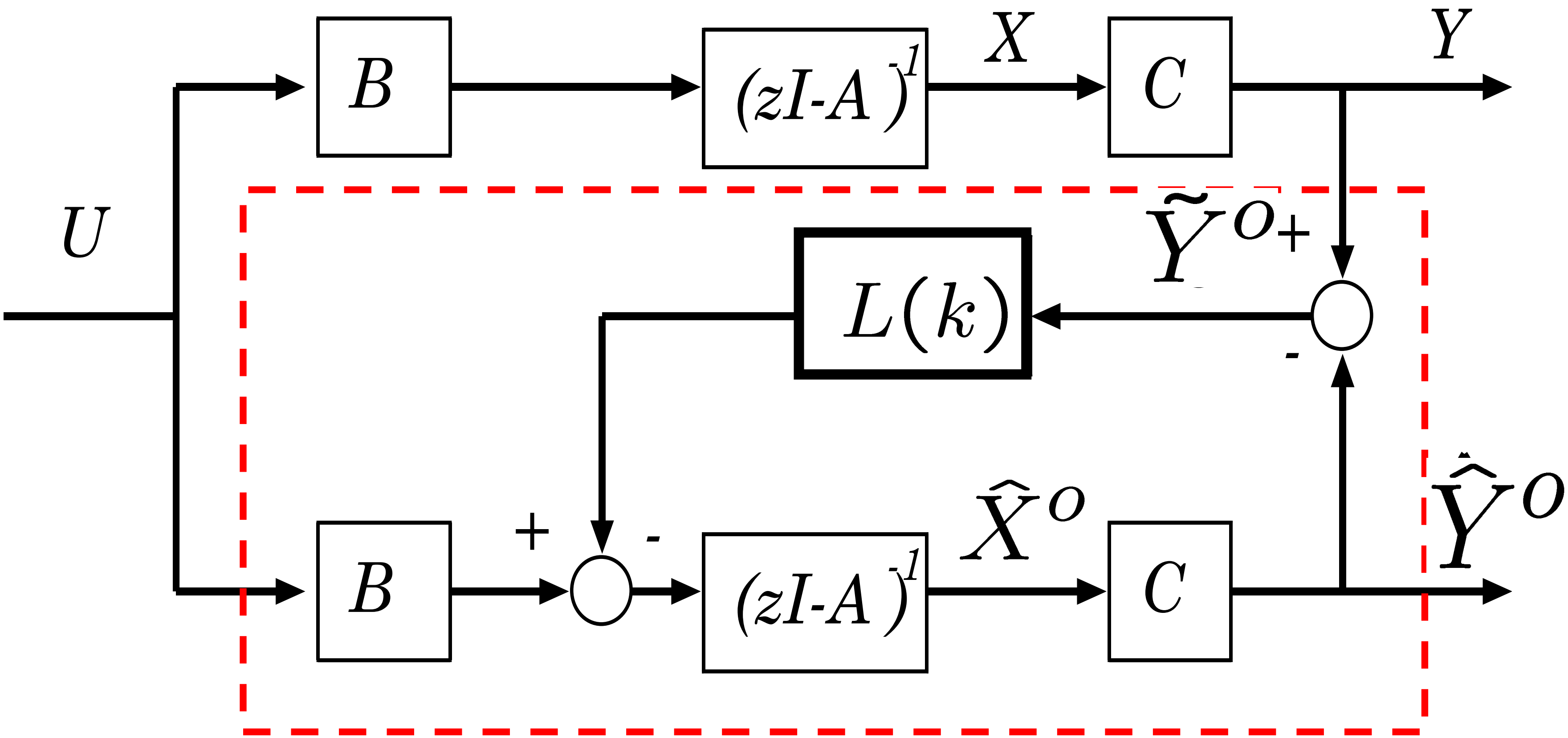
A-priori state observer structure:

$$
\begin{aligned}
\hat{x}^o(k+1) &= A\,\hat{x}^o(k) + B\,u(k) + L(k)\,\tilde{y}^o(k) \\
\tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k)
\end{aligned}
$$

$$
\begin{aligned}
L(k) &= A\,M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1} \\
M(k+1) &= AM(k)A^T + B_wW(k)B_w^T \\
&\quad - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1}CM(k)A^T \\
M(0) &= X_o
\end{aligned}
$$

Kalman Filter Solution V-2

- Same structure as deterministic a-priori observer



Kalman Filter Solution V-1 (Review)

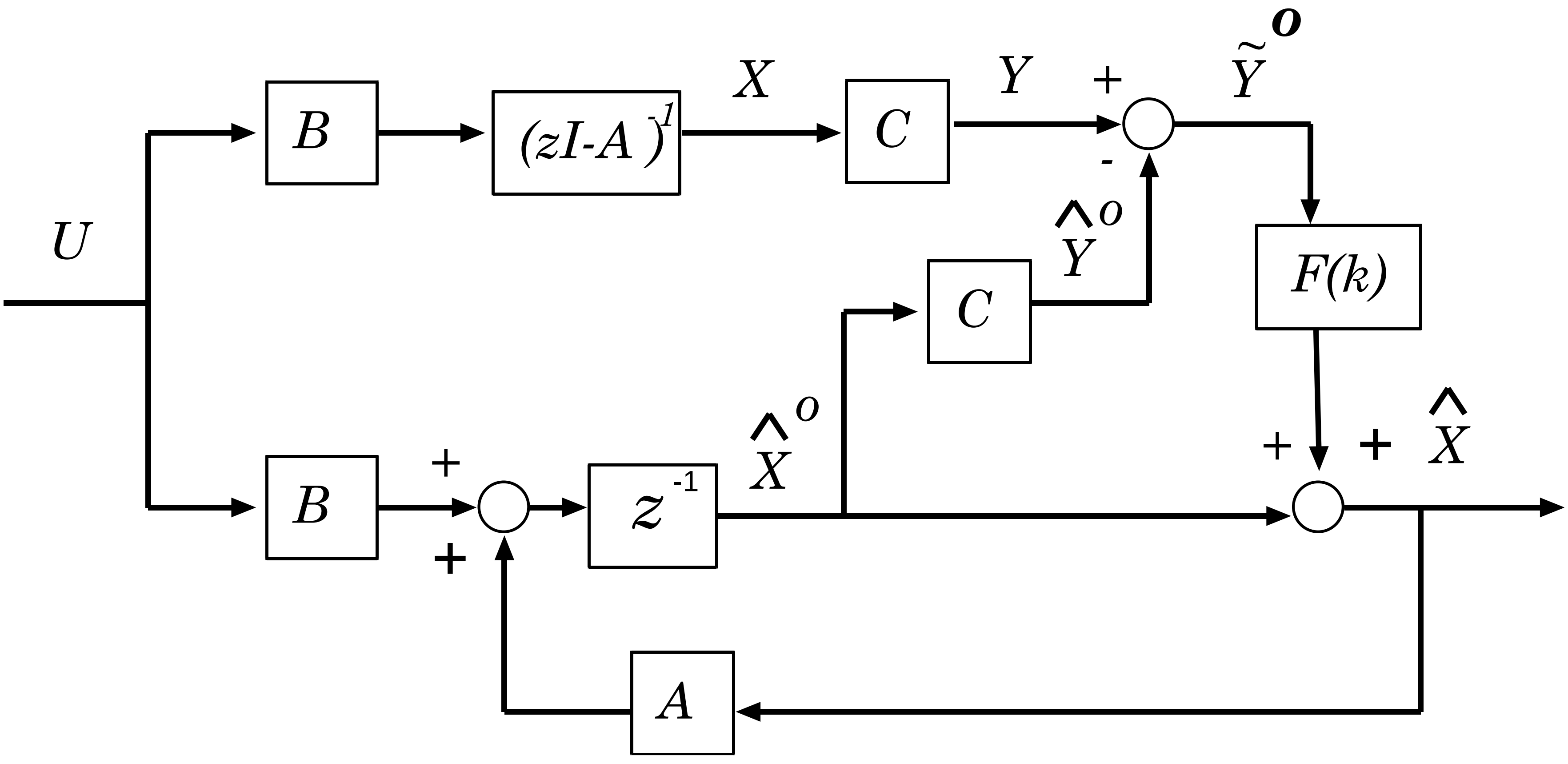
A-posteriori state observer structure:

$$
\begin{aligned}
\widehat{x}(k) &= \widehat{x}^o(k) + F(k)\,\widetilde{y}^o(k) \\
\widehat{x}^o(k+1) &= A\,\widehat{x}(k) + B\,u(k) \\
\widetilde{y}^o(k) &= y(k) - C\,\widehat{x}^o(k)
\end{aligned}
$$

$$
\begin{aligned}
F(k) &= M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1} \\
M(k+1) &= AM(k)A^T + B_wW(k)B_w^T \\
&\quad - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T
\end{aligned}
$$

Kalman Filter Solution V-1

- A-posteriori estimator as output



Kalman Filter, State Space Form

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)\tilde{y}^o(k)$$

$$\hat{x}(k) = \hat{x}^o(k) + F(k)\tilde{y}^o(k)$$

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

$$\hat{x}^o(k+1) = [A - L(k)C]\hat{x}^o(k) + Bu(k) + L(k)y(k)$$

$$\hat{x}(k) = [I - F(k)C]\hat{x}^o(k) + F(k)y(k)$$

Kalman Filter, State Space Form

$$\hat{x}^o(k+1) = [A - L(k)C]\hat{x}^o(k) + \begin{bmatrix} B & L(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\hat{x}(k) = [I - F(k)C]\hat{x}^o(k) + \begin{bmatrix} 0 & F(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\begin{aligned}
F(k) &= M(k)C^T \left[C\,M(k)C^T + V(k)\right]^{-1} \\
L(k) &= AM(k)C^T \left[C\,M(k)C^T + V(k)\right]^{-1} \\
M(k+1) &= AM(k)A^T + B_w W(k) B_w^T \\
&\quad - AM(k)C^T \left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T
\end{aligned}$$

Kalman Filter (KF) Properties

- The KF is a linear **time varying** estimator, even when the system is LTI and the noises are WSS
- The KF is the **optimal state estimator** when the input and measurement noises are Gaussian.
- The KF is still the **optimal *linear* state estimator** even when the input and measurement noises are **not** Gaussian.

Kalman Filter (KF) Properties

The KF a-priori output error (*a-priori output residual*)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is often called the ***innovation***

it contains only the "new information" in $y(k)$

Moreover,

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,j) = [CM(k)C^T + V(k)]\delta(j)$$

i.e. $\tilde{y}^o(k)$ is an uncorrelated RVS

Kalman Filter (KF) Properties

Proof: It suffices to show that

$$E\{\tilde{y}^o(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

By causality, $E\{v(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$

By least squares property 1,

$$E\{\tilde{x}^o(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

$$\Longrightarrow E\{[C\tilde{x}^o(k) + v(k)]\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

$$\Longrightarrow E\{\tilde{y}^o(k)\tilde{y}^{oT}(j)\} = 0 \qquad j < k$$

$\blacksquare$

KF as an innovations filter

We will assume, without loss of generality that the control input is zero, i.e.

$$u(k) = 0 \qquad\qquad k = 0,\ 1,\ \cdots$$

- **Plant:**

$$\begin{aligned} x(k+1) &= A\,x(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

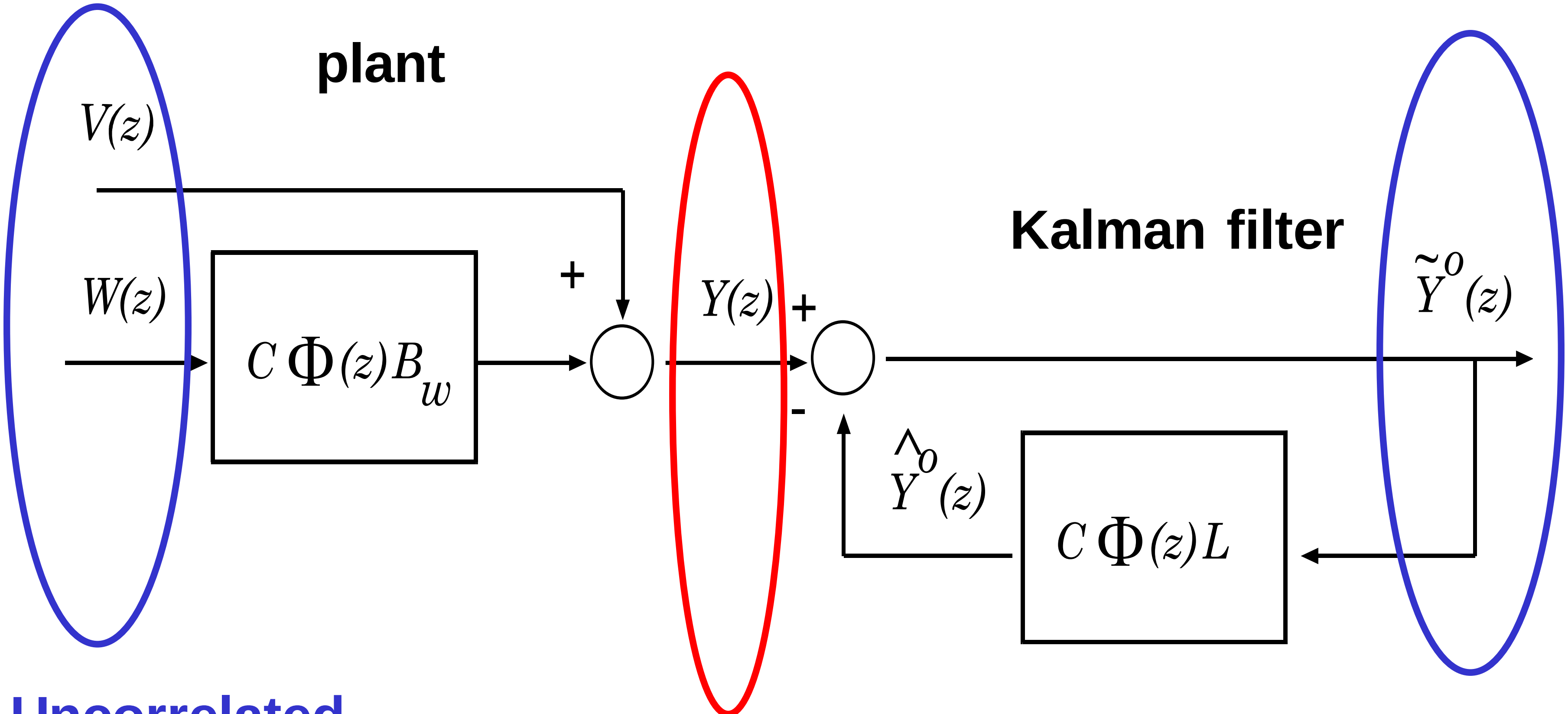
- **Kalman filter V-2:**

$$\widehat{x}^o(k+1) = A\widehat{x}^o(k) + L(k)\tilde{y}^o(k)$$

$$\widehat{y}^o(k) = C\widehat{x}^o(k)$$

KF as an innovations filter

$$\Phi(z) = (zI - A)^{-1}$$



Uncorrelated noise input

$$\begin{bmatrix} V(k) & 0 \\ 0 & W(k) \end{bmatrix}$$

Correlated noise output

Uncorrelated noise output

$$CM(k)C^T + V(k)$$