ME 233 Advanced Control II

Lecture 5
Random Vector Sequences

(ME233 Class Notes pp. PR6-PR10)

Outline

- Random vector sequences
  - Mean, auto-covariance, cross-covariance
- MIMO Linear Time Invariant Systems
- State space systems driven by white noise
- Lyapunov equation for covariance propagation

Random vector sequences

A two-sided random vector sequence is a collection of random vectors

$$X = \{\cdots \; X(-1), \; X(0), \; X(1),$$

$$X(2), \; X(3), \; \cdots \}$$

each $X(k) \in \mathcal{R}^n$ is itself a random vector

defined over the same probability space $(\Omega, \, \mathcal{S}, \, P)$

Random vector sequences

We either will use

$$\{X(k)\}_{k=-\infty}^{\infty} \qquad \text{or} \qquad X(k)$$

Shorthand (sloppy) notation

to denote the two-sided random vector sequence.

Each element $X(k)$ of the sequence is a random vector:

$$X(k) : \Omega \to \mathcal{R}^n$$

Random vector sequences

A ***sample sequence***

corresponds to the value of

$$\{\ldots,\ X(-1),\ X(0),\ X(1),\ X(2),\ldots\}$$

obtained after performing an experiment

2^{nd} order statistics

For a two-sided **Random Vector Sequence (RVS)**

$$\{X(k)\}_{k=-\infty}^{\infty}$$

Expected value or mean of *X(k)*,

$$\boxed{E\{X(k)\} = m_X(k) \in \mathcal{R}^n}$$

Auto-covariance

Define: $$\tilde{X}(k) = X(k) - m_X(k)$$

$$\boxed{\Lambda_{XX}(k, \underline{j}) = E\left\{\tilde{X}(k+\underline{j})\tilde{X}^T(k)\right\}}$$

$$\Lambda_{XX}(k, \underline{j}) = E\left\{\begin{bmatrix} \tilde{X}_1(k+\underline{j}) \\ \vdots \\ \tilde{X}_n(k+\underline{j}) \end{bmatrix}\begin{bmatrix} \tilde{X}_1(k) & \cdots & \tilde{X}_n(k) \end{bmatrix}\right\}$$

Cross-covariance

Define: $$\tilde{X}(k) = X(k) - m_X(k)$$

$$\tilde{Y}(k) = Y(k) - m_Y(k)$$

$$\boxed{\Lambda_{XY}(k, j) = E\left\{\tilde{X}(k+j)\tilde{Y}^T(k)\right\}}$$

$$\Lambda_{XY}(k, j) = E\left\{\begin{bmatrix} \tilde{X}_1(k+j) \\ \vdots \\ \tilde{X}_n(k+j) \end{bmatrix} \begin{bmatrix} \tilde{Y}_1(k) & \cdots & \tilde{Y}_n(k) \end{bmatrix}\right\}$$

Wide Sense Stationary (WSS)

A two-sided random vector sequence $\{X(k)\}_{k=-\infty}^{\infty}$

is **WSS** if:

1) $E\{X(k)\} = m_X$ (time invariant)

2) $\Lambda_{XX}(\underline{k}, l) = \Lambda_{XX}(\underline{k+M}, l)\}$

(only depends on l)

Auto-covariance function

For WSS RVS, the auto-covariance is only a function of the correlation index j

$$\Lambda_{XX}(j) = E\left\{\tilde{X}(k+j)\tilde{X}^T(k)\right\}$$

for ***any*** index k

$$\Lambda_{XX}(l) = \Lambda^T_{XX}(-l)$$

Auto-covariance function Z-transform

$$\Lambda_{XX}(l) = \Lambda_{XX}^{T}(-l)$$

$$\Big\Downarrow \quad \textit{Z-transform}$$

$$\hat{\Lambda}_{XX}(z) = \hat{\Lambda}_{XX}^{T}(z^{-1})$$

Auto-covariance function

$$\Lambda_{XX}(l) = \Lambda_{XX}^{T}(-l)$$

Proof:

$$\begin{aligned}\Lambda_{XX}^{T}(-l) &= E\{(X(k-l)-m_X)(X(k)-m_X)^T\}^T \\ &= E\{(X(k)-m_X)(X(k-l)-m_X)^T\}\end{aligned}$$

Define $\bar{k} := k - l$

$$\begin{aligned}\Lambda_{XX}^{T}(-l) &= E\{(X(\bar{k}+l)-m_X)(X(\bar{k})-m_X)^T\} \\ &= \Lambda_{XX}(l)\end{aligned}$$

■

Auto-covariance function Z-transform

$$\hat{\Lambda}_{XX}(z) = \hat{\Lambda}_{XX}^{T}(z^{-1})$$

Proof:

$$\hat{\Lambda}_{XX}^{T}(z^{-1}) = \left[\sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) z^{+l}\right]^{T} = \sum_{l=-\infty}^{\infty} \Lambda_{XX}^{T}(l) z^{+l}$$

Define $n := -l$

$$\hat{\Lambda}_{XX}^{T}(z^{-1}) = \sum_{n=-\infty}^{\infty} \Lambda_{XX}^{T}(-n) z^{-n} = \sum_{n=-\infty}^{\infty} \Lambda_{XX}(n) z^{-n}$$

$$= \hat{\Lambda}_{XX}(z)$$

$\blacksquare$

Cross-covariance function

$X(k)$ and $Y(k)$

are two **WSS** random vector sequences

$$\boxed{\Lambda_{XY}(j) = E\{\tilde{X}(k+j)\tilde{Y}^T(k)\}}$$

for ***any*** index k

Notice that:

$$\boxed{\Lambda_{XY}(l) = \Lambda_{YX}^T(-l)}$$

Auto-covariance function Z-transform

$$\Lambda_{XY}(l) = \Lambda_{YX}^{T}(-l)$$

$$\Big\Downarrow \quad \textit{Z-transform}$$

$$\hat{\Lambda}_{XY}(z) = \hat{\Lambda}_{YX}^{T}(z^{-1})$$

Cross-covariance function

$$\Lambda_{XY}(l) = \Lambda_{YX}^{T}(-l)$$

Proof:

$$\begin{aligned}\Lambda_{YX}^{T}(-l) &= E\{(Y(k-l)-m_Y)(X(k)-m_X)^T\}^T \\ &= E\{(X(k)-m_X)(Y(k-l)-m_Y)^T\}\end{aligned}$$

Define $\bar{k} := k - l$

$$\begin{aligned}\Lambda_{YX}^{T}(-l) &= E\{(X(\bar{k}+l)-m_X)(Y(\bar{k})-m_Y)^T\} \\ &= \Lambda_{XY}(l)\end{aligned}$$

$\blacksquare$

Auto-covariance function Z-transform

$$\hat{\Lambda}_{XY}(z) = \hat{\Lambda}^T_{YX}(z^{-1})$$

Proof:

$$\hat{\Lambda}^T_{YX}(z^{-1}) = \left[\sum_{l=-\infty}^{\infty} \Lambda_{YX}(l) z^{+l}\right]^T = \sum_{l=-\infty}^{\infty} \Lambda^T_{YX}(l) z^{+l}$$

Define $n := -l$

$$\hat{\Lambda}^T_{YX}(z^{-1}) = \sum_{n=-\infty}^{\infty} \Lambda^T_{YX}(-n) z^{-n} = \sum_{n=-\infty}^{\infty} \Lambda_{XY}(n) z^{-n}$$

$$= \hat{\Lambda}_{XY}(z)$$

$\blacksquare$

Ergodicity

A **Wide Sense Stationary** random sequence

is **ergodic**

if its ensemble average = time average (constant)

$$E\{X(k)\} = m_X$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x(k)$$

with probability 1
(almost surely)

sample sequence

Ergodicity

For any **WSS ergodic** random sequence

we can approximate the covariance as a "time average"

$$\Lambda_{XX}(j) = \qquad E\{\,\tilde{X}(k+j)\,\tilde{X}^T(k)\,\}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \tilde{x}(k+j)\tilde{x}^T(k)$$

with probability 1
(almost surely)

$$\tilde{x}(k) = x(k) - m_X$$

sample sequence

Power Spectral Density Function

Fourier transform of the auto-covariance function:

$$\Phi_{XX}(\omega) = \mathcal{F}\{\Lambda_{XX}(\cdot)\}$$

$$= \sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) e^{-j\omega l}$$

Complex-valued matrix

l: correlation index

Note:

The power spectral density function is periodic, with period $T = 2\pi$

$$e^{-j\omega l} = \cos(\omega l) - j\sin(\omega l)$$

Power Spectral Density Function

Using the inverse Fourier transform we obtain:

$$\Lambda_{XX}(l) = \mathcal{F}^{-1}\{\Phi_{XX}(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega l}\Phi_{XX}(\omega)\, d\omega$$

Power Spectral Density Function

Properties of the power spectral density function:

1. $\Phi_{XX}(\omega) = \Phi^T_{XX}(-\omega)$

2. $\Phi_{XX}(\omega) = \Phi^*_{XX}(\omega) \qquad \omega \in [-\pi, \pi]$

3. $\Phi_{XX}(\omega) \succeq 0 \qquad \omega \in [-\pi, \pi]$

4. $\Lambda_{XX}(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \Phi_{XX}(\omega)\, d\omega$

Power Spectral Density Function

1. $$\Phi_{XX}(\omega) = \Phi^T_{XX}(-\omega) \qquad \omega \in [-\pi, \pi]$$

Proof:

$$\Phi^T_{XX}(-\omega) = \left[\sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) e^{j\omega l}\right]^T = \sum_{l=-\infty}^{\infty} \Lambda^T_{XX}(l) e^{j\omega l}$$

Define $n := -l$

$$\Phi^T_{XX}(-\omega) = \sum_{n=-\infty}^{\infty} \Lambda^T_{XX}(-n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \Lambda_{XX}(n) e^{-j\omega n}$$
$$= \Phi_{XX}(\omega)$$

$\blacksquare$

Power Spectral Density Function

2. $$\Phi_{XX}(\omega) = \Phi^*_{XX}(\omega) \qquad \omega \in [-\pi, \pi]$$

Proof:

$$\Phi^*_{XX}(\omega) = \left[\sum_{l=-\infty}^{\infty} \Lambda_{XX}(l) e^{-j\omega l}\right]^* = \sum_{l=-\infty}^{\infty} \Lambda^T_{XX}(l) e^{j\omega l}$$

Define $n := -l$

$$\Phi^*_{XX}(\omega) = \sum_{n=-\infty}^{\infty} \Lambda^T_{XX}(-n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \Lambda_{XX}(n) e^{-j\omega n}$$
$$= \Phi_{XX}(\omega)$$

■

Power Spectral Density Function

Properties of the power spectral density function:
(scalar case)

1. $$\Phi_{XX}(\omega) = \Phi_{XX}(-\omega)$$

2. $$\Phi_{XX}(\omega) \text{ is real} \qquad \omega \in [-\pi, \pi]$$

3. $$\Phi_{XX}(\omega) \geq 0 \qquad \omega \in [-\pi, \pi]$$

4. $$\Lambda_{XX}(0) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{XX}(\omega)\, d\omega$$

White noise vector sequence

A **WSS** random vector sequence $\{W(k)\}_{k=-\infty}^{\infty}$ **is white** if:

$$\boxed{\Lambda_{WW}(l) = \Sigma_{WW}\,\delta(l)}$$

where

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW} = E\{\tilde{W}(k)\tilde{W}^T(k)\} \qquad \tilde{W}(k) = W(k) - m_W$$

$$\Sigma_{WW} = \Sigma_{WW}^T \succeq 0$$

White noise vector sequence

Given the white WSS random sequence $\{W(k)\}_{k=-\infty}^{\infty}$

with

$$\Lambda_{WW}(l) = \Sigma_{WW}\,\delta(l)$$

Its power spectral density (Fourier transform)

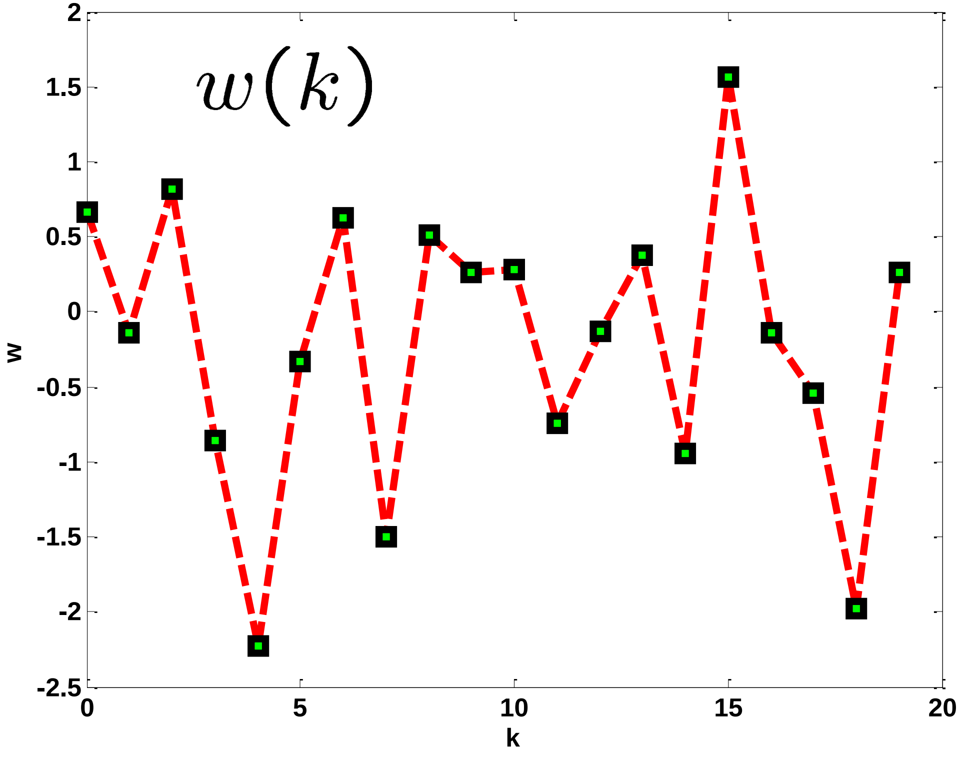
$$\Phi_{WW}(\omega) = \sum_{l=-\infty}^{\infty} \Lambda_{WW}(l) e^{-j\omega l}$$

is

$$\boxed{\Phi_{WW}(\omega) = \Sigma_{WW}}$$

White noise illustration (scalar case)

- zero-mean white noise $W(k)$



first 20 samples

Matlab commands:

w = randn(N,1);

MIMO Linear Time Invariant Systems

Let $\{g(k)\}_{k=-\infty}^{\infty}$ with $g(k) \in \mathcal{R}^{p \times m}$

be the pulse response of an asymptotically stable MIMO LTI system

Transfer function

$$G(z) = \mathcal{Z}\{g(k)\} = \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

MIMO Linear Time Invariant Systems

Let $U(k) \in \mathcal{R}^m$ be WSS

The forced response (zero initial state)

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$

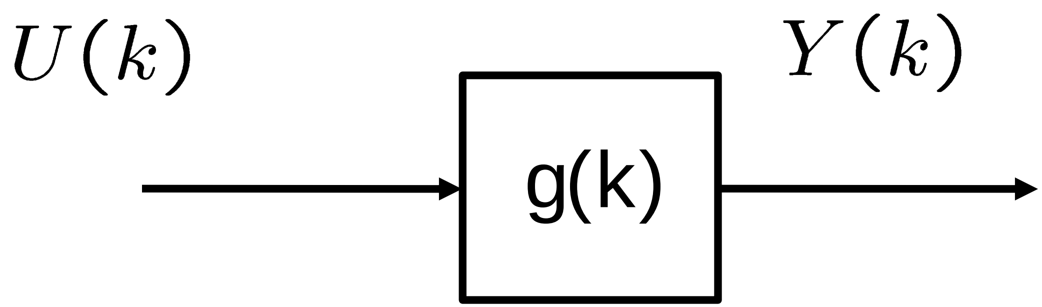
$$g(k) \in \mathcal{R}^{p \times m}$$

$\boxed{Y(k) \in \mathcal{R}^p \quad \text{is also WSS}}$

MIMO Linear Time Invariant Systems

Let $U(k) \in \mathcal{R}^m$ be WSS

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i) U(k-i)$$



MIMO Linear Time Invariant Systems

We will assume

$\{U(k)\}_{k=-\infty}^{\infty}$ is zero mean, I.e.

$$E\left\{U(k)\right\} = m_U = 0$$

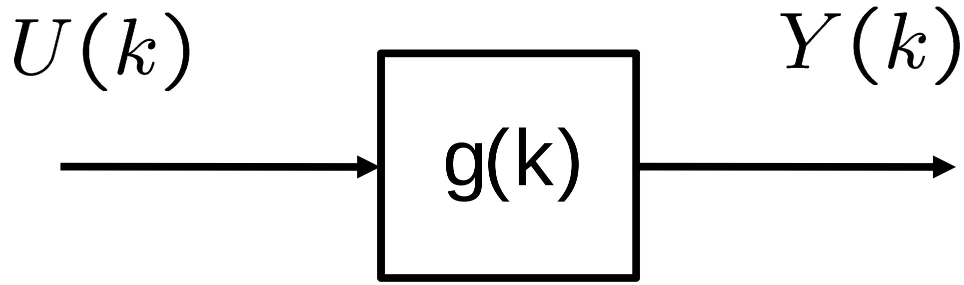
Thus, the forced response output is also zero mean

$$E\left\{Y(k)\right\} = m_Y = 0$$

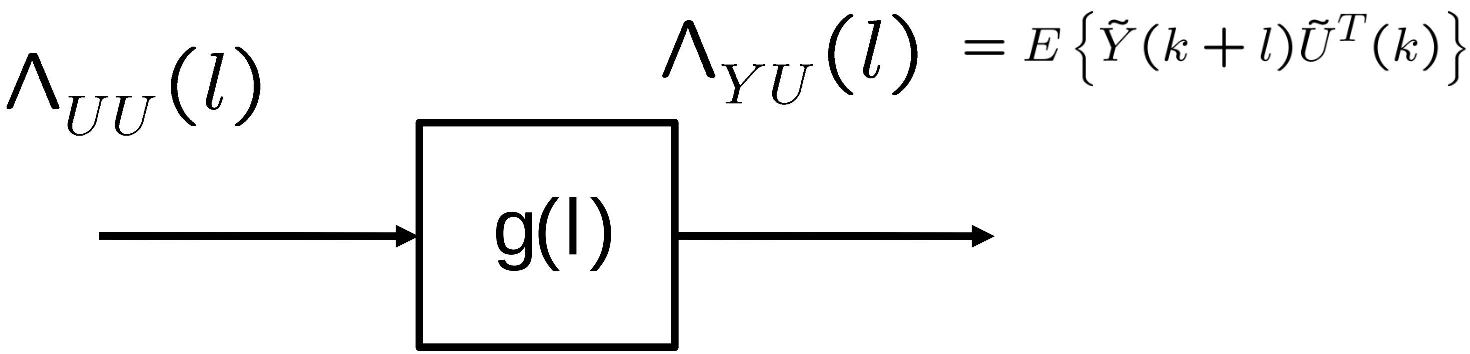
MIMO Linear Time Invariant Systems

Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS

If



Then:



MIMO Linear Time Invariant Systems

$$\boxed{\Lambda_{YU}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UU}(l-i)}$$

Proof:

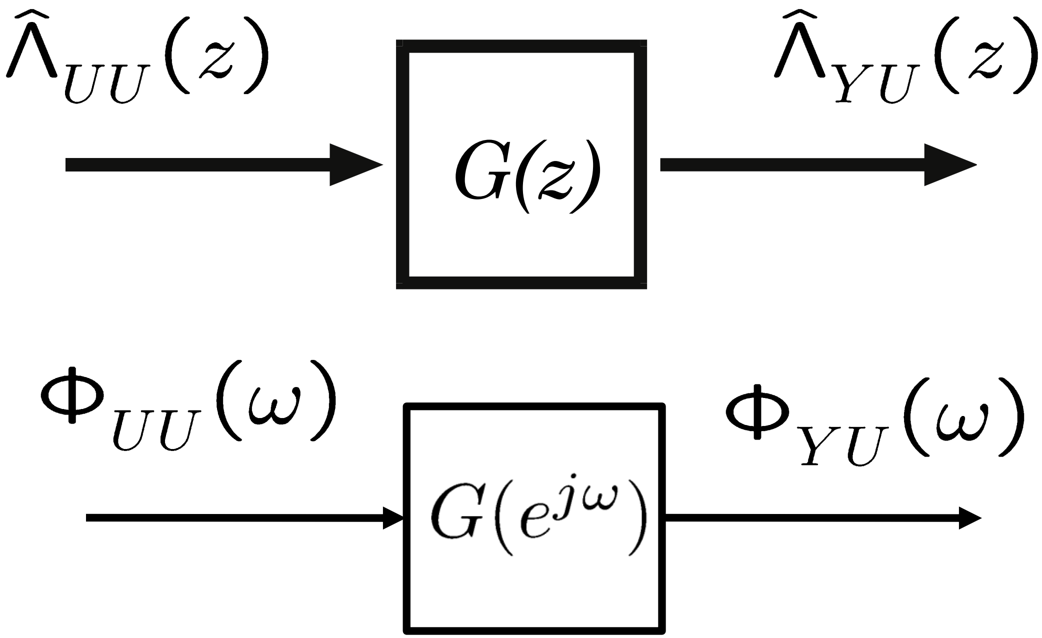
$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i) \qquad (m_U = 0)$$

Then:

$$\begin{aligned}
\Lambda_{YU}(l) &= E\{Y(k+l)U^T(k)\} \\
&= E\left\{\left[\sum_{i=-\infty}^{\infty} g(i)\,U(k+l-i)\right]U^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} g(i)\,E\left\{U(k+l-i)U^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UU}(l-i) \qquad \blacksquare
\end{aligned}$$

MIMO Linear Time Invariant Systems

Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS



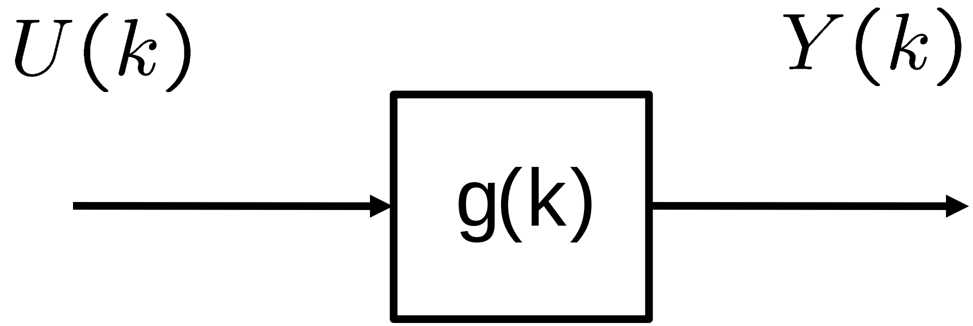
$$\hat{\Lambda}_{UU}(z) = \mathcal{Z}\{\Lambda_{UU}(l)\} \qquad \hat{\Lambda}_{YU}(z) = \mathcal{Z}\{\Lambda_{YU}(l)\}$$

$$\Phi_{UU}(\omega) = \hat{\Lambda}_{UU}(z)\Big|_{z=e^{j\omega}} \qquad \Phi_{YU}(\omega) = \hat{\Lambda}_{YU}(z)\Big|_{z=e^{j\omega}}$$

MIMO Linear Time Invariant Systems

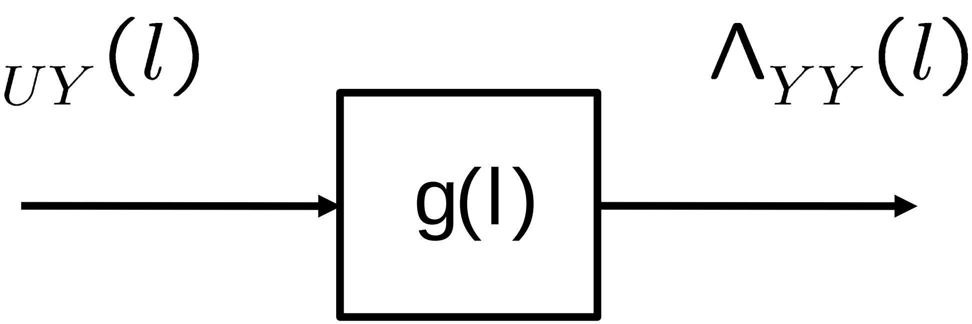
Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS

If



Then:

$E\left\{\tilde{U}(k+l)\tilde{Y}^T(k)\right\} = \Lambda_{UY}(l)$



MIMO Linear Time Invariant Systems

$$\boxed{\Lambda_{YY}(l) = \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UY}(l-i)}$$

Proof:

$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i) \qquad (m_U = 0)$$

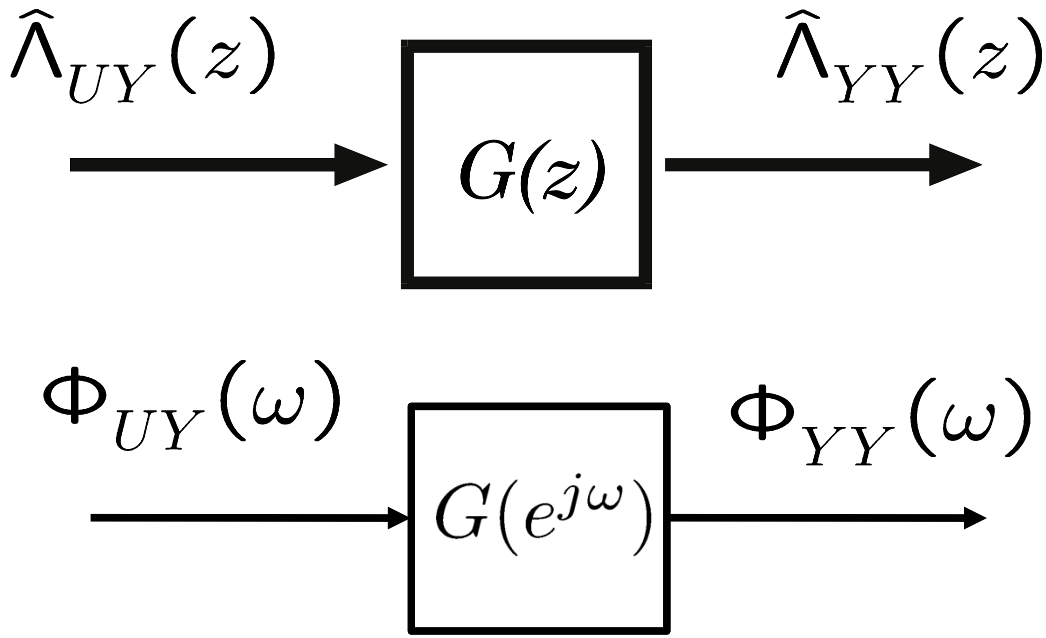
Then:

$$\begin{aligned}
\Lambda_{YY}(l) &= E\{Y(k+l)Y^T(k)\} \\
&= E\left\{\left[\sum_{i=-\infty}^{\infty} g(i)\,U(k+l-i)\right] Y^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} g(i)\,E\left\{U(k+l-i)Y^T(k)\right\} \\
&= \sum_{i=-\infty}^{\infty} g(i)\,\Lambda_{UY}(l-i)
\end{aligned}$$

$\blacksquare$

MIMO Linear Time Invariant Systems

Let $\{U(k)\}_{k=-\infty}^{\infty}$ be WSS



$$\widehat{\Lambda}_{UY}(z) = \mathcal{Z}\{\Lambda_{UY}(l)\} \qquad \widehat{\Lambda}_{YY}(z) = \mathcal{Z}\{\Lambda_{YY}(l)\}$$

$$\Phi_{UY}(\omega) = \widehat{\Lambda}_{UY}(z)\Big|_{z=e^{j\omega}} \qquad \Phi_{YY}(\omega) = \widehat{\Lambda}_{YY}(z)\Big|_{z=e^{j\omega}}$$

MIMO Linear Time Invariant Systems

$$\boxed{\Phi_{UY}(\omega) = \Phi^T_{YU}(-\omega)}$$

This is a consequence of the fact that

$$\Lambda_{UY}(l) = \Lambda^T_{YU}(-l)$$

MIMO Linear Time Invariant Systems

$$\Phi_{UY}(\omega) = \Phi_{YU}^T(-\omega)$$

Proof:

$$\Phi_{YU}^T(-\omega) = \left[\sum_{l=-\infty}^{\infty} \Lambda_{YU}(l) e^{j\omega l}\right]^T = \sum_{l=-\infty}^{\infty} \Lambda_{YU}^T(l) e^{j\omega l}$$

Define $n := -l$

$$\begin{aligned} \Phi_{YU}^T(-\omega) &= \sum_{n=-\infty}^{\infty} \Lambda_{YU}^T(-n) e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \Lambda_{UY}(n) e^{-j\omega n} = \Phi_{UY}(\omega) \end{aligned}$$

$\blacksquare$

MIMO Linear Time Invariant Systems

If
$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$

Then:

$$\boxed{\widehat{\Lambda}_{YY}(z) = G(z)\,\widehat{\Lambda}_{UU}(z)\,G^T(z^{-1})}$$

MIMO Linear Time Invariant Systems

$$\widehat{\Lambda}_{YY}(z) = G(z)\,\widehat{\Lambda}_{UU}(z)\,G^T(z^{-1})$$

Proof:

$$\begin{aligned}
\widehat{\Lambda}_{YY}(z) &= G(z)\widehat{\Lambda}_{UY}(z) \\
&= G(z)\left[\widehat{\Lambda}_{YU}(z^{-1})\right]^T \\
&= G(z)\left[G(z^{-1})\widehat{\Lambda}_{UU}(z^{-1})\right]^T \\
&= G(z)\widehat{\Lambda}_{UU}^T(z^{-1})G^T(z^{-1}) \\
&= G(z)\widehat{\Lambda}_{UU}(z)G^T(z^{-1})
\end{aligned}$$

MIMO Linear Time Invariant Systems

If
$$Y(k) = \sum_{i=-\infty}^{\infty} g(i)U(k-i)$$

Then:

$$\boxed{\Phi_{YY}(\omega) = G(e^{j\omega})\,\Phi_{UU}(\omega)\,G^*(e^{j\omega})}$$

MIMO Linear Time Invariant Systems

$$\Phi_{YY}(\omega) = G(e^{j\omega})\,\Phi_{UU}(\omega)\,G^*(e^{j\omega})$$

Proof:

$$\widehat{\Lambda}_{YY}(z) = G(z)\,\widehat{\Lambda}_{UU}(z)\,G^T(z^{-1})$$

Let $z = e^{j\omega}$

$$\widehat{\Lambda}_{YY}(e^{j\omega}) = G(e^{j\omega})\,\widehat{\Lambda}_{UU}(e^{j\omega})\,G^T(e^{-j\omega})$$

$$\Phi_{YY}(\omega) \qquad\qquad \Phi_{UU}(\omega) \qquad G^*(e^{j\omega})$$

Next Topic

- Stable causal LTI systems driven by <u>uncorrelated</u> random vector sequences

  - *Similar to "white"*
  - *Definition in 2 slides*

- State-space

- No WSS assumption

2nd order statistics of a random sequence

We now consider one-sided random sequence

$$\{X(k)\}_{k=0}^{\infty}$$

Expected value or mean of $X(k)$,

$$E\{X(k)\} = m_X(k)$$

Auto-covariance function:

$$\Lambda_{XX}(k, j) =$$
$$E\left\{\left[X(k+j) - m_X(k+j)\right]\left[X(k) - m_X(k)\right]^T\right\}$$

Uncorrelated random vector sequence

A random vector sequence $\{W(k)\}_{k=-\infty}^{\infty}$ is **uncorrelated** if:

$$\boxed{\Lambda_{WW}(k,l) = \Sigma_{WW}(k)\,\delta(l)}$$

where

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW}(k) = E\{\tilde{W}(k)\tilde{W}^T(k)\}$$

$$\tilde{W}(k) = W(k) - m_W(k)$$

$$\Sigma_{WW}(k) = \Sigma_{WW}^T(k) \succeq 0$$

Subtracting the mean

- Define

$$\tilde{X}(k) = X(k) - m_X(k)$$

Auto-covariance

$$\Lambda_{XX}(k, j) = E\left\{\tilde{X}(k+j)\,\tilde{X}^T(k)\right\}$$

State space systems

Consider a LTI system driven by an uncorrelated RVS:

$$
\begin{aligned}
X(k+1) &= A\,X(k) + B\,W(k) \\
Y(k) &= C\,X(k)
\end{aligned}
$$

$X(k) \in \mathcal{R}^n$

$W(k) \in \mathcal{R}^p$

$Y(k) \in \mathcal{R}^m$

State space systems

$W(k)$ is an uncorrelated RVS

$$m_W(k) = E\{W(k)\}$$

$$\Lambda_{WW}(k,l) = \Sigma_{WW}(k)\,\delta(l)$$

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$\Sigma_{WW}(k) = E\{\tilde{W}(k)\tilde{W}^T(k)\} \in \mathcal{R}^{p\times p}$$

State space systems

$$
\begin{aligned}
X(k+1) &= A\,X(k) + B\,W(k) \\
Y(k) &= C\,X(k)
\end{aligned}
$$

State Initial Conditions (IC):

$$m_X(0) = E\{X(0)\}$$

$$\Lambda_{XX}(0,0) \;=\; E\left\{\tilde{X}(0)\tilde{X}^T(0)\right\}$$

$$E\left\{\tilde{X}(0)\tilde{W}^T(k)\right\} = 0, \qquad \forall k \geq 0$$

Dynamics of the mean

$$
\begin{aligned}
X(k+1) &= A\,X(k) + B\,W(k) \\
Y(k) &= C\,X(k)
\end{aligned}
$$

Taking expectations on the equations above:

$$
m_X(k+1) = A\,m_X(k) + B\,m_W(k)
$$

$$
m_Y(k) = C\,m_X(k)
$$

State space systems

Subtracting the means we obtain,

$$\tilde{X}(k+1) \;=\; A\,\tilde{X}(k) + B\,\tilde{W}(k)$$

$$\tilde{Y}(k) \;=\; C\,\tilde{X}(k)$$

Where now

$$m_{\tilde{W}}(k) = 0$$

$$m_{\tilde{X}}(k) = 0$$

Causality in cross-covariance

$$E\{\tilde{W}(k+j)\tilde{X}^T(k)\} = 0 \qquad \forall j \geq 0, k \geq 0$$

Proof: (by induction on k)

1. Base case, $k=0$: trivial by assumptions on system

2. Case $k>0$:

$$E\{\tilde{W}(k+j)\tilde{X}^T(k)\} = E\{\tilde{W}(k+j)[A\tilde{X}(k-1) + B\tilde{W}(k-1)]^T\}$$

$$= E\{\tilde{W}(k+j)\tilde{X}^T(k-1)\}A^T + \cancelto{0}{E\{\tilde{W}(k+j)\tilde{W}^T(k-1)\}B^T}$$

$$= 0$$

(by induction hypothesis)

Covariance propagation

$$\tilde{X}(k+1) \;=\; A\,\tilde{X}(k) + B\,\tilde{W}(k)$$

Notice that:

$$\tilde{X}(k+1)\tilde{X}^T(k+1) =$$

$$\left[A\,\tilde{X}(k) + B\,\tilde{W}(k)\right]\left[A\,\tilde{X}(k) + B\,\tilde{W}(k)\right]^T$$

Covariance propagation

Taking expectations to:

$$
\begin{aligned}
\underbrace{\tilde{X}(k+1)\tilde{X}^T(k+1)}_{\Lambda_{XX}(k+1,0)} &= A\,\tilde{X}(k)\tilde{X}^T(k)\,A^T \\
&+ A\,\tilde{X}(k)\tilde{W}^T(k)\,B^T \\
&+ B\,\tilde{W}(k)\tilde{X}^T(k)\,A^T \\
&+ B\,\tilde{W}(k)\tilde{W}^T(k)\,B^T
\end{aligned}
$$

Covariance propagation

Notice that:

$$
\begin{aligned}
\Lambda_{XX}(k+1,0) &= A\,\Lambda_{XX}(k,0)\,A^T \\
&+ \cancel{A\,\Lambda_{XW}(k,0)\,B^T} \; \mathbf{0} \\
&+ \cancel{B\,\Lambda_{WX}(k,0)\,A^T} \; \mathbf{0} \\
&+ B\,\Lambda_{WW}(k,0)\,B^T
\end{aligned}
$$

(***W*(*k*)** is an uncorrelated RVS)

$$
\begin{aligned}
\Lambda_{XW}(k,0) &= \Lambda^T_{WX}(k,0) \\
&= E\left\{\tilde{X}(k)\tilde{W}^T(k)\right\} = 0
\end{aligned}
$$

Covariance propagation

We obtain the following Lyapunov equation:

$$\boxed{\Lambda_{XX}(k+1,0) \;=\; A\,\Lambda_{XX}(k,0)\,A^T + B\,\Sigma_{WW}(k)\,B^T}$$

$$\Lambda_{XX}(k,0) \;=\; E\left\{\tilde{X}(k)\tilde{X}^T(k)\right\}$$

$$\Lambda_{WW}(k,0) \;=\; E\left\{\tilde{W}(k)\tilde{W}^T(k)\right\} = \Sigma_{WW}(k)$$

Covariance propagation

From the output equation

$$\tilde{Y}(k) \;=\; C\,\tilde{X}(k)$$

we obtain

$$\Lambda_{YY}(k,0) \;=\; C\,\Lambda_{XX}(k,0)\,C^T$$

Covariance propagation

Lets now compute,

$$\Lambda_{XX}(k,l) \;=\; E\left\{\tilde{X}(k+l)\tilde{X}^T(k)\right\} \qquad l \geq 0$$

Using the solution of the LTI system,

$$\tilde{X}(k+l) \;=\; A^l\,\tilde{X}(k) + \sum_{j=k}^{k+l-1} A^{k+l-1-j} B\,\tilde{W}(j)$$

Covariance propagation

$$\tilde{X}(k+l) \;=\; A^l\,\tilde{X}(k) + \sum_{j=k}^{k+l-1} A^{k+l-1-j} B\,\tilde{W}(j)$$

$$\Lambda_{XX}(k,l) \;=\; E\left\{\tilde{X}(k+l)\tilde{X}^T(k)\right\}$$

$$= \; A^l\,E\{\tilde{X}(k)\tilde{X}^T(k)\}$$

$$+ \sum_{j=k}^{k+l-1} A^{k+l-1-j} B\,\underbrace{\cancel{E\{\tilde{W}(j)\tilde{X}^T(k)\}}}_{\boldsymbol{0}}$$

$$= A^l\Lambda_{XX}(k,0)$$

Covariance propagation

Lets now compute

$$\Lambda_{XX}(k,-l) = E\left\{\tilde{X}(k-l)\tilde{X}^T(k)\right\} \qquad l \geq 0$$

$$= E\{\tilde{X}(k)\tilde{X}^T(k-l)\}^T$$

define $\hat{k} := k - l$

$$\Lambda_{XX}(k,-l) = E\{\tilde{X}(\hat{k}+l)\tilde{X}^T(\hat{k})\}^T$$

$$= \Lambda_{XX}^T(\hat{k},l) = \left[A^l \Lambda_{XX}(k-l,0)\right]^T$$

$$= \Lambda_{XX}(k-l,0)(A^l)^T$$

Covariance propagation

$$\Lambda_{XX}(k,l) \;=\; E\left\{\tilde{X}(k+l)\tilde{X}^T(k)\right\}$$

Satisfies:

$$\boxed{\begin{array}{lr} \Lambda_{XX}(k,l) \;=\; A^l\,\Lambda_{XX}(k,0) & l \geq 0 \\[2em] \Lambda_{XX}(k,-l) = \Lambda_{XX}(k-l,0)\,(A^l)^T & l \geq 0 \end{array}}$$

Stationary covariance equation

If $W(k)$ is WSS

and A is Schur (i.e. all eigenvalues inside unit circle):

and $X(k)$ and $Y(k)$ will converge to WSS RVS:

$$\lim_{k\to\infty} m_X(k) = \bar{m}_X \qquad \lim_{k\to\infty} m_Y(k) = C\bar{m}_X$$

$$\lim_{k\to\infty} \Lambda_{XX}(k,0) = \bar{\Lambda}_{XX}(0) \qquad \lim_{k\to\infty} \Lambda_{YY}(k,0) = \bar{\Lambda}_{YY}(0)$$

$$= C\bar{\Lambda}_{XX}(0)C^T$$

WSS Stationary covariance equation

For $W(k)$ WSS, and A Schur,

$$m_X(k+1) \; = \; A\, m_X(k) + B\, m_W$$

converges to

$$\bar{m}_X \; = \; [I - A]^{-1} B\, m_W$$

WSS Stationary covariance equation

For $W(k)$ WSS, and A Schur,

$$\bar{\Lambda}_{XX}(0) = \lim_{k\rightarrow\infty} E\{\tilde{X}(k)\tilde{X}^T(k)\}$$

Satisfies the Lyapunov equation:

$$\boxed{A\,\bar{\Lambda}_{XX}(0)\,A^T - \bar{\Lambda}_{XX}(0) = -B\,\Sigma_{WW}\,B^T}$$

WSS Stationary covariance equation

For $W(k)$ WSS, and A Schur,

$$\bar{\Lambda}_{XX}(l) = \lim_{k\to\infty} E\{\tilde{X}(k+l)\tilde{X}^T(k)\}$$

Satisfies

$$\boxed{\begin{aligned} \bar{\Lambda}_{XX}(l) &= A^l\,\bar{\Lambda}_{XX}(0) \qquad\qquad l \geq 0 \\ \bar{\Lambda}_{XX}(-l) &= \bar{\Lambda}_{XX}(0)(A^l)^T \end{aligned}}$$

Illustration – first order system

- Plant:

$$Y(k+1) = 0.5\,Y(k) + 1\,W(k)$$

- Input:

$$m_W(k) = 1 \qquad \Lambda_{WW}(k,l) = 0.2\,\delta(l)$$

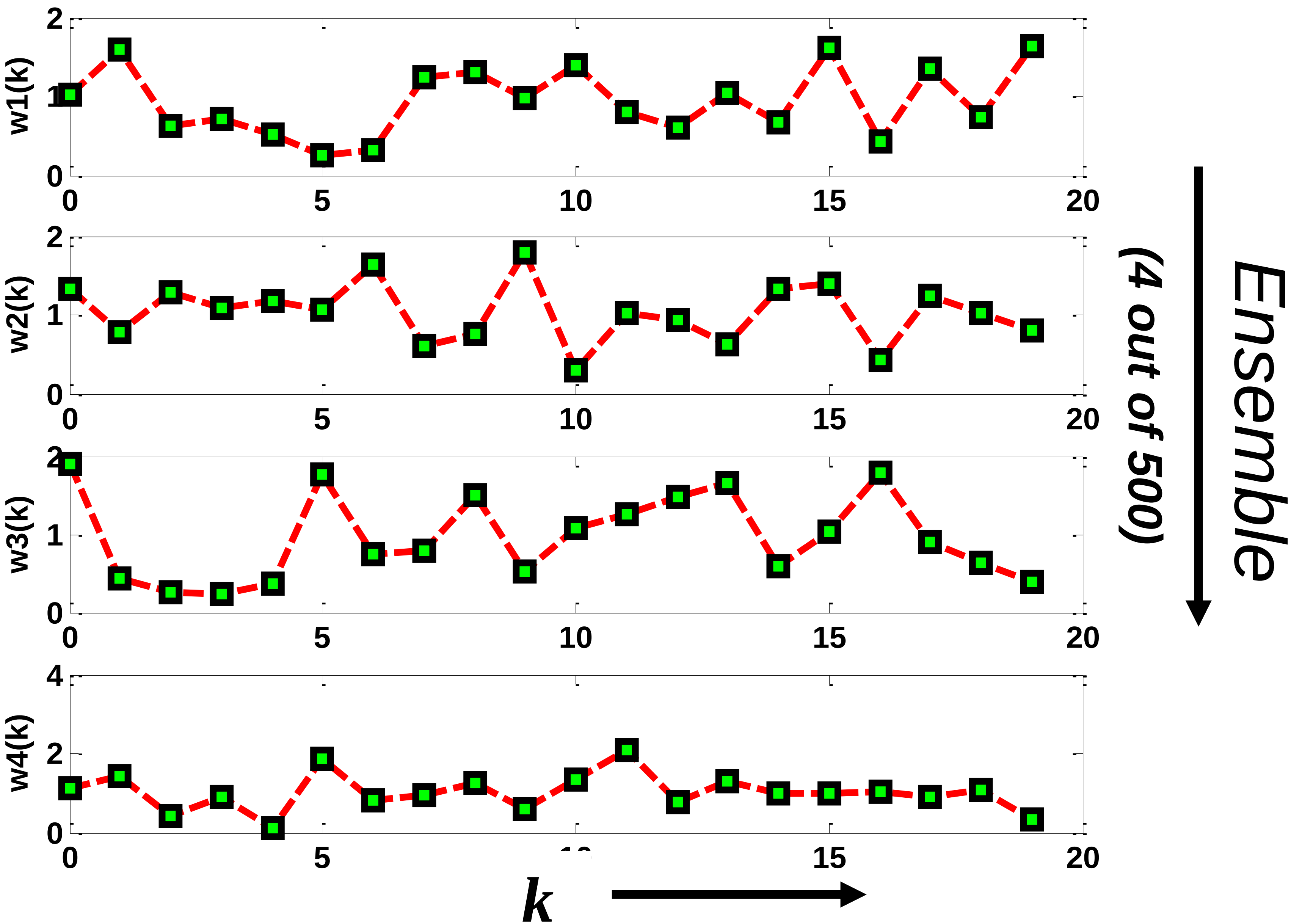
- State initial conditions:

$$m_Y(0) = 0 \qquad \Lambda_{YY}(0,0) = .1$$

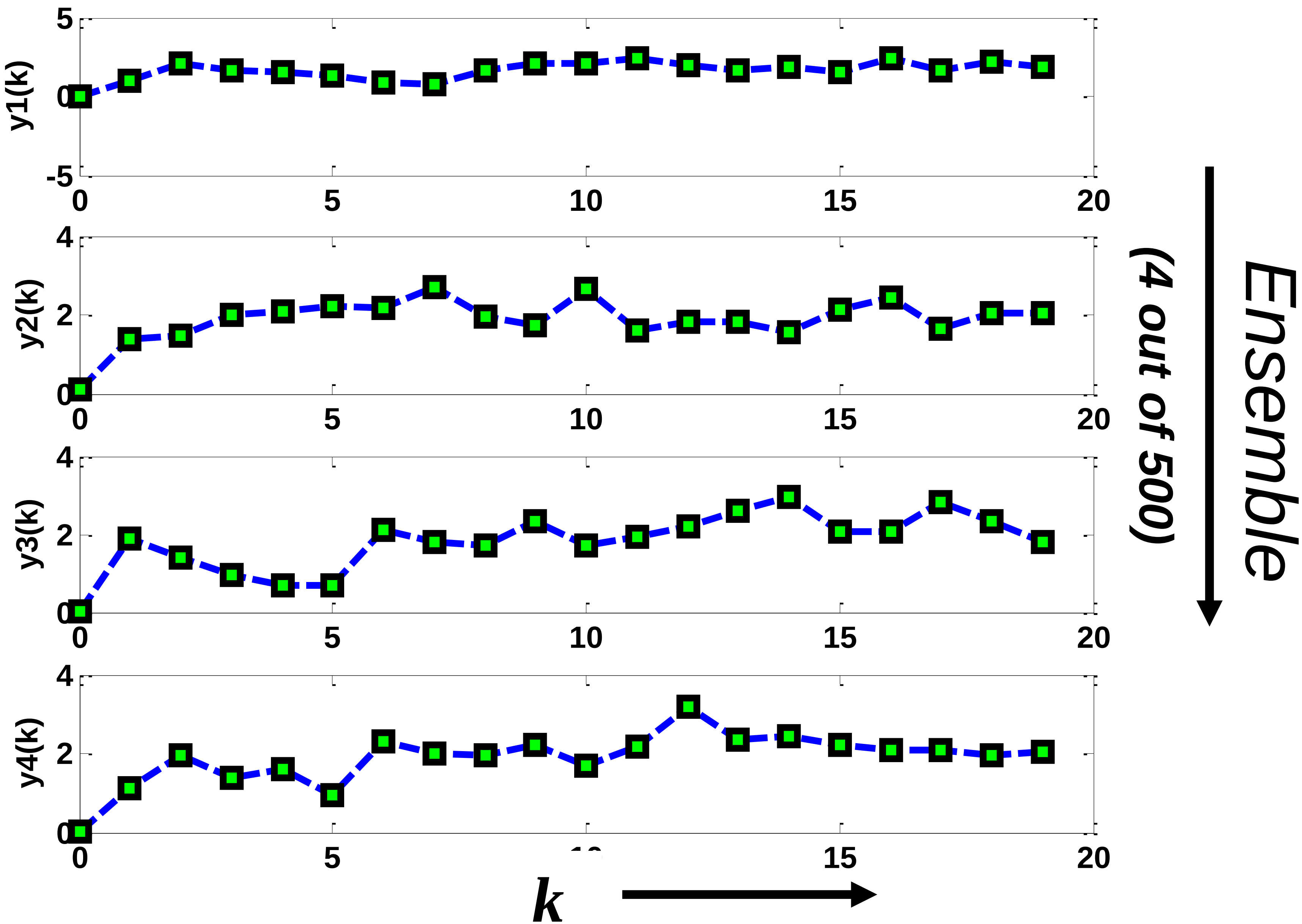
Matlab simulation: 500 sample sequences

```
lyy0  = 0.1
lww = 0.2
sys1=ss(.5,1,1,0,1)
N=20;
p=500;
w = sqrt(lww)*randn(N,p)+1;
y = zeros(N,p);
y0 = sqrt(lyy0)*randn(1,p);
k = (0:1:N-1)';

for j=1:p
    [y(:,j),k] = lsim(sys1,w(:,j),k,y0(1,j));
end

m_y=mean(y')
L_yy=diag(cov(y'));
```

$$\boldsymbol{W(k)} \quad m_W(k) = 1 \quad \Lambda_{WW}(k,l) = 0.2\,\delta(l)$$



Y(k) $m_Y(0) = 0 \quad \Lambda_{YY}(0,0) = .1$



Mean Transient Response

Actual:

$$m_Y(k+1) = 0.5\, m_Y(k) + 1$$

$$m_Y(0) = 0$$

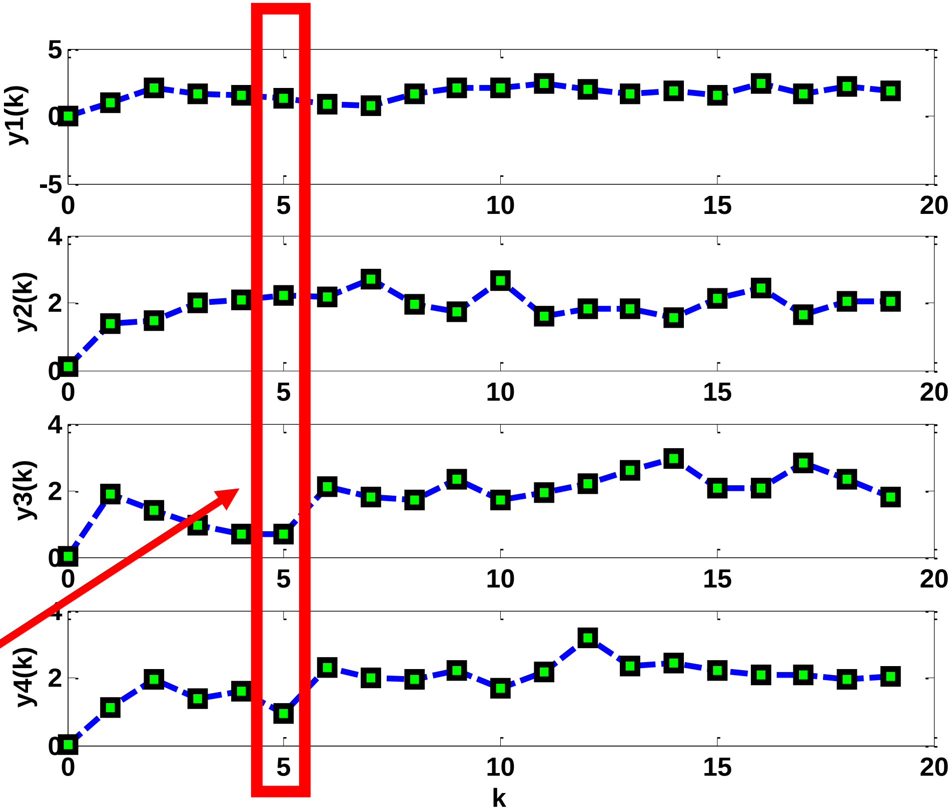
Matlab calculation:

Ensemble mean

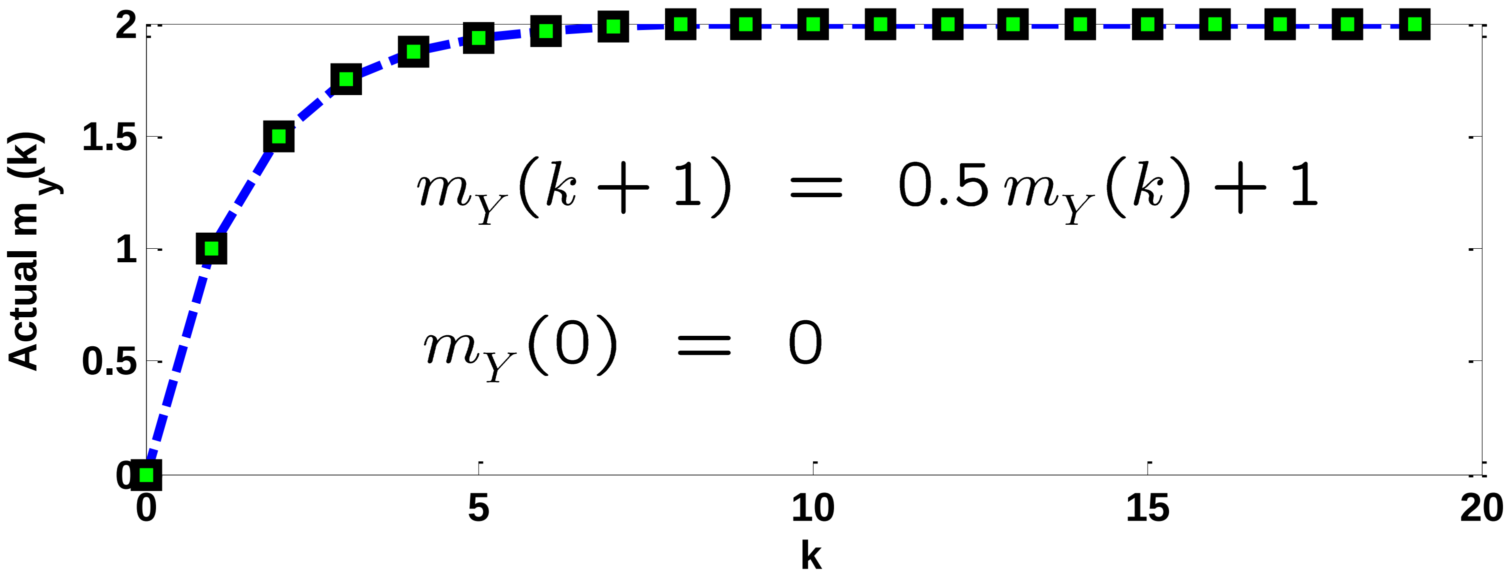
m_y=mean(y');

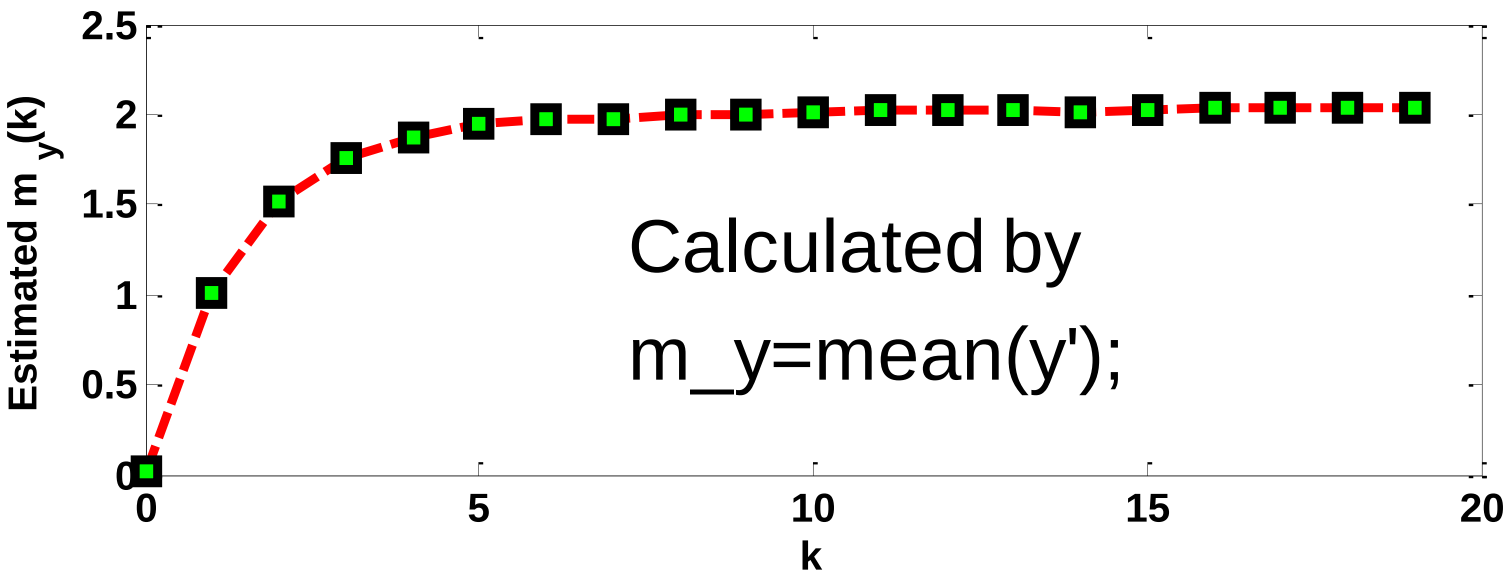
Not using ergodicity because not WSS!!

$$\approx m_Y(5)$$



Mean Transient Response





Covariance Transient Response

Actual:

$$\Lambda_{XX}(k+1,0) = 0.5^2\,\Lambda_{XX}(k,0) + 0.2$$

$$\Lambda_{XX}(0,0) = .1$$

$\Lambda_{WW}(k,l) = 0.2\,\delta(l)$

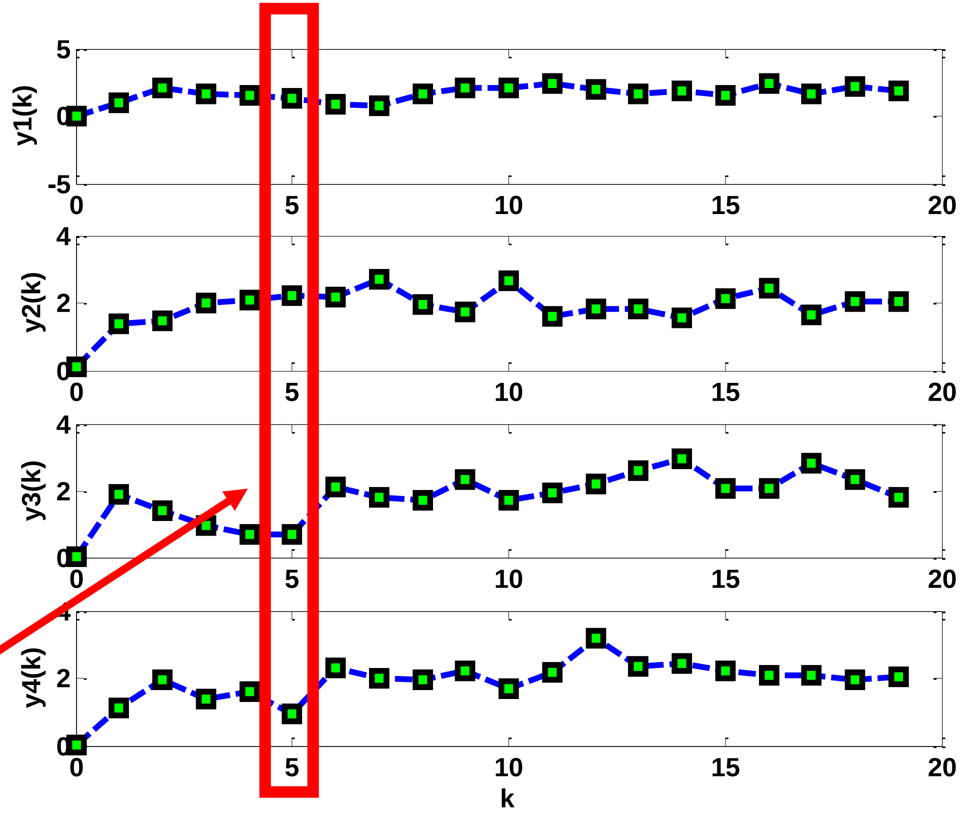
Matlab calculation:

Ensemble covariance

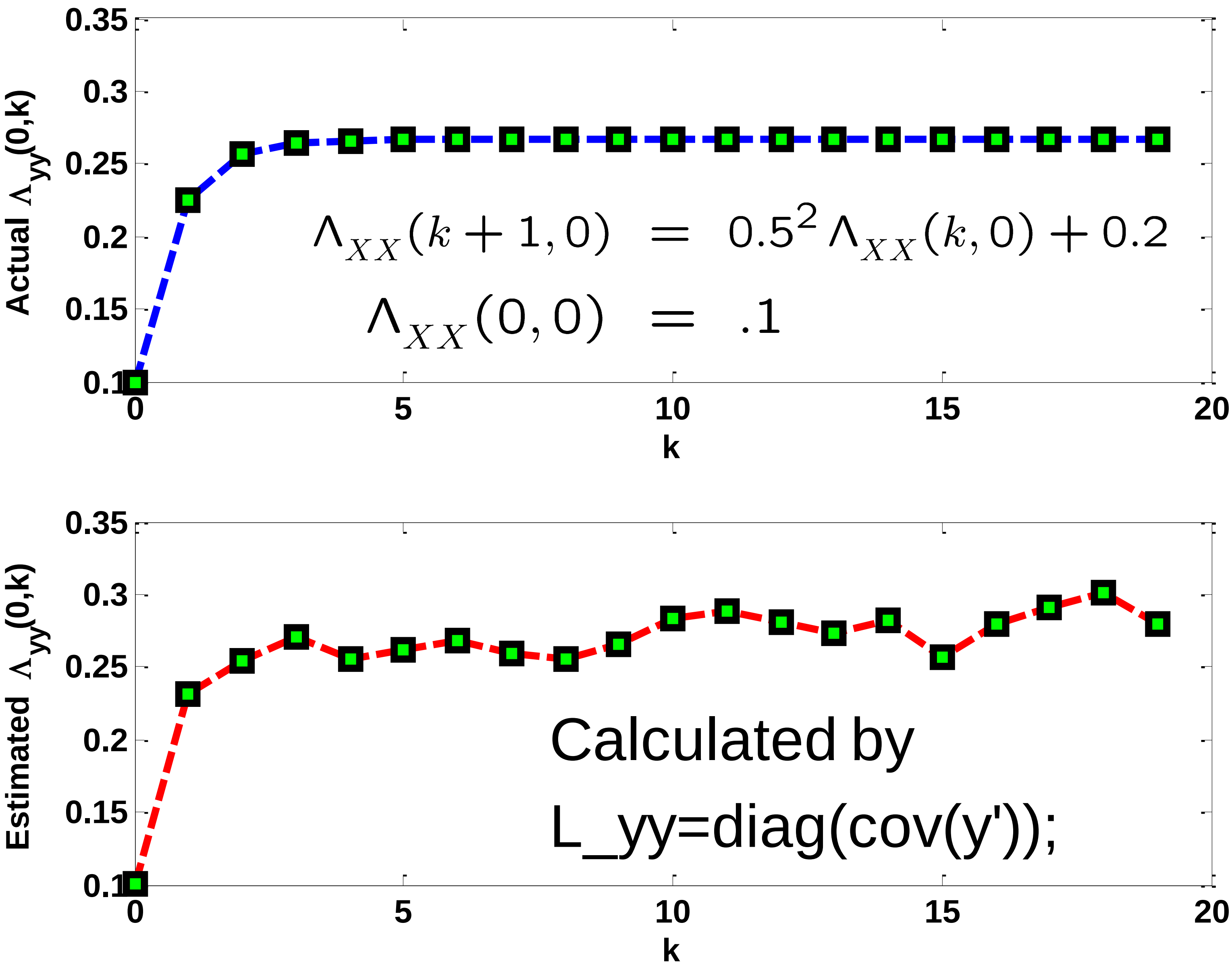
L_yy=diag(cov(y'));

Not using ergodicity because not WSS!!

$$\approx \Lambda_{YY}(5,0)$$



Covariance Transient Response



Steady State Covariance