ME 233 Advanced Control II

Lecture 3
Introduction to Probability Theory

(ME233 Class Notes pp. PR1-PR3)

Outline

- Continuous random variable
- CDF, PDF, expectation and variance
- Uniform and normal PDFs

Continuous random variable

A continuous-valued random X variable takes on a range of **real** values

- For the probability space, $(\Omega, \mathcal{S}, P)$
- A random variable X is a mapping $X : \Omega \to \mathcal{R}$

Example:

- An experiment whose outcome is a real number, e.g. measurement of a noisy voltage.

$$X \in [V_{\min}, V_{\max}]$$

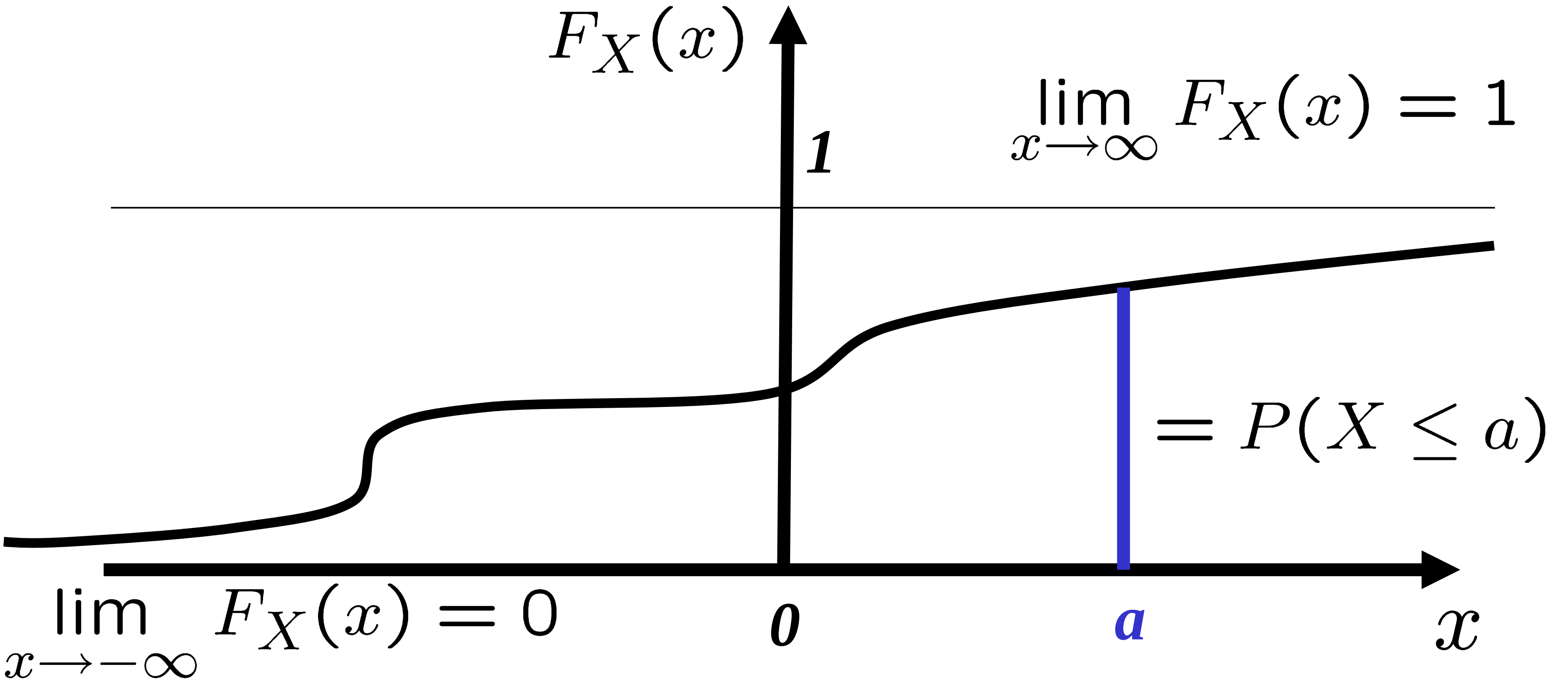
Cumulative Distribution Function

Cumulative distribution function (CDF) associated with the random variable $\boldsymbol{X}$:

$$\boxed{F_X(x) = P(X \leq x)}$$

The probability that the random variable $\boldsymbol{X}$ will be less than or equal to the value $\boldsymbol{x}$

Properties of the cumulative distribution



Properties of the cumulative distribution

$$F_X(x) = P(X \leq x)$$

1. $\lim_{x \to -\infty} F_X(x) = 0$

2. $\lim_{x \to \infty} F_X(x) = 1$

3. $F_X(x)$ is a monotone non decreasing

4. $F_X(x)$ is left-continuous

Probability Density Function

For a ***differentiable*** cumulative distribution function,

$$F_X(x) = P(X \leq x)$$

Define the **probability density function (PDF)**,

$$\boxed{p_X(x) = \frac{dF_X(x)}{dx}}$$

Probability Density Function

$$p_X(x) = \frac{dF_X(x)}{dx}$$

Interpretation:

$$p_X(x)\,\Delta x \approx P(x \leq X \leq x + \Delta x)$$

for small Δx

Loosely interpret this as the probability that X takes a value close to x

Probability Density Function

$$p_X(x) = \frac{dF_X(x)}{dx}$$

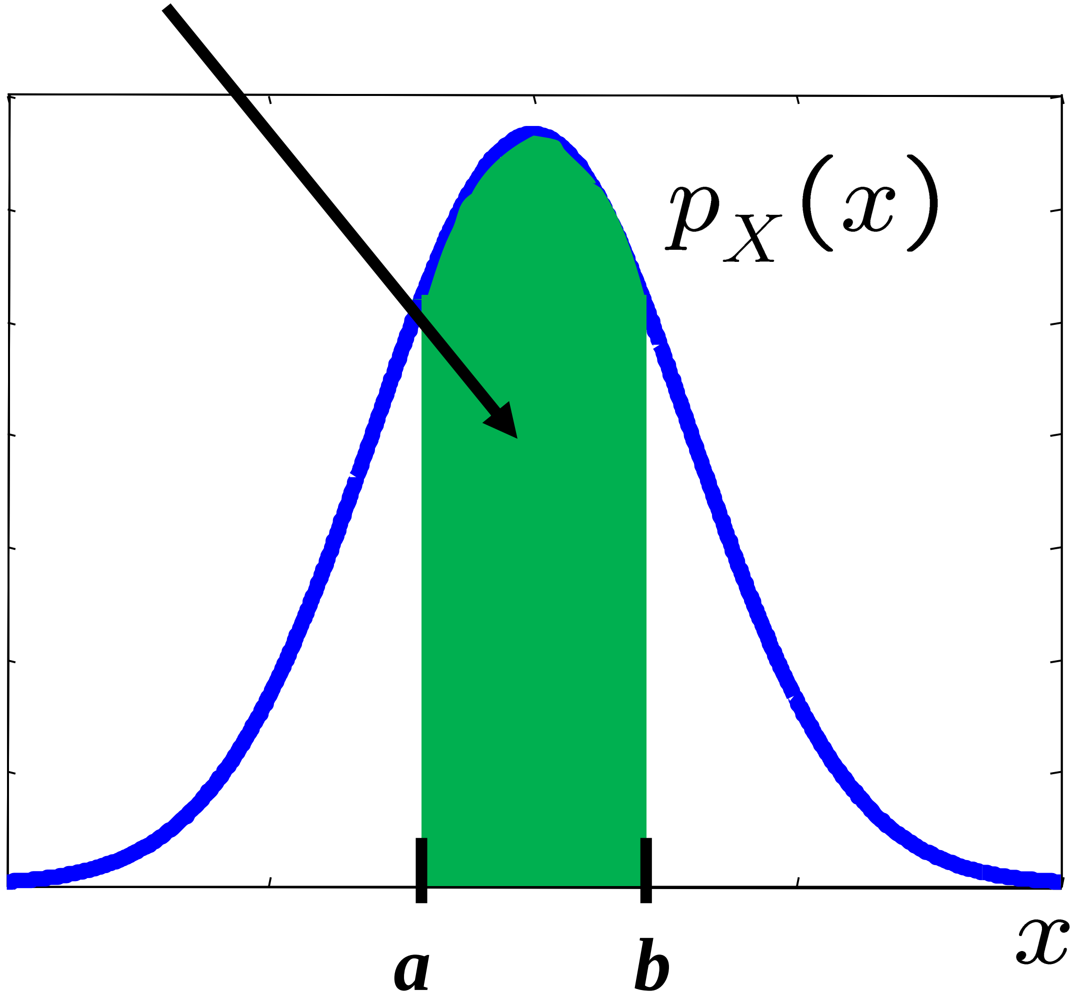
By the Fundamental Theorem of Calculus

$$\int_a^b p_X(x)dx = F_X(b) - F_X(a)$$

$$\Rightarrow \boxed{\int_a^b p_X(x)dx = P(a \leq X \leq b)}$$

Probability Density Function

$$\int_a^b p_X(x)dx = P(a \le X \le b)$$



Probability Density Function

Property:

$$\int_{-\infty}^{\infty} p_X(x)dx = 1$$

because

$$\int_{-\infty}^{\infty} p_X(x)dx = P(-\infty \leq X \leq \infty)$$

Expectation

The ***expected value*** of random variable X is:

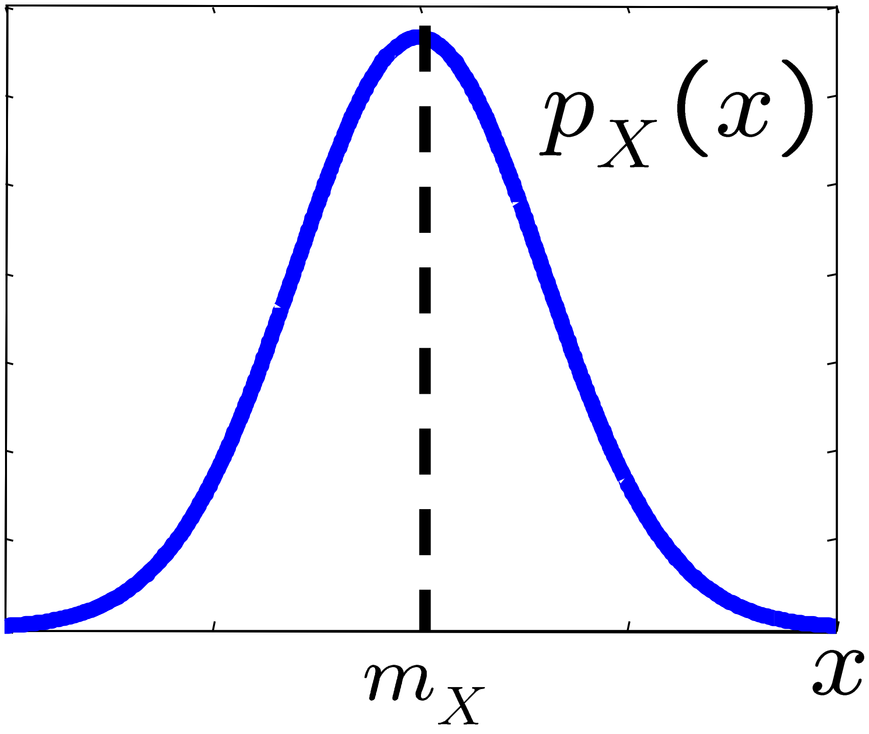
$$\boxed{E[X] = \int_{-\infty}^{\infty} x\, p_X(x)\, dx}$$

This is the average value of X.

It is also called the ***mean*** of X
or the ***first moment*** of X

Expected value - notation

$$\boxed{m_X = \widehat{x} = E[X]}$$



Expected value of a function

f : real valued function of random variable X

$$Y = f(X)$$

The expected value of Y is

$$\boxed{E[Y] = \int_{-\infty}^{\infty} f(x)\, p_X(x)\, dx}$$

Variance

The ***variance*** of random variable X is:

$$\boxed{\begin{aligned} \sigma_X^2 &= E[(X - m_X)^2] \\ &= \int_{-\infty}^{\infty} (x - m_X)^2 \, p_X(x) \, dx \end{aligned}}$$

where $m_X = E[X]$

σ_X Is called the standard deviation of X

Variance

$$\begin{aligned}\sigma_X^2 &= E[(X - m_X)^2] \\ &= E[X^2] - m_X^2\end{aligned}$$

where

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \, p_X(x) \, dx$$

Proof

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - m_X)^2 \, p_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left(x^2 - 2xm_X + m_X^2\right) p_X(x)\, dx$$

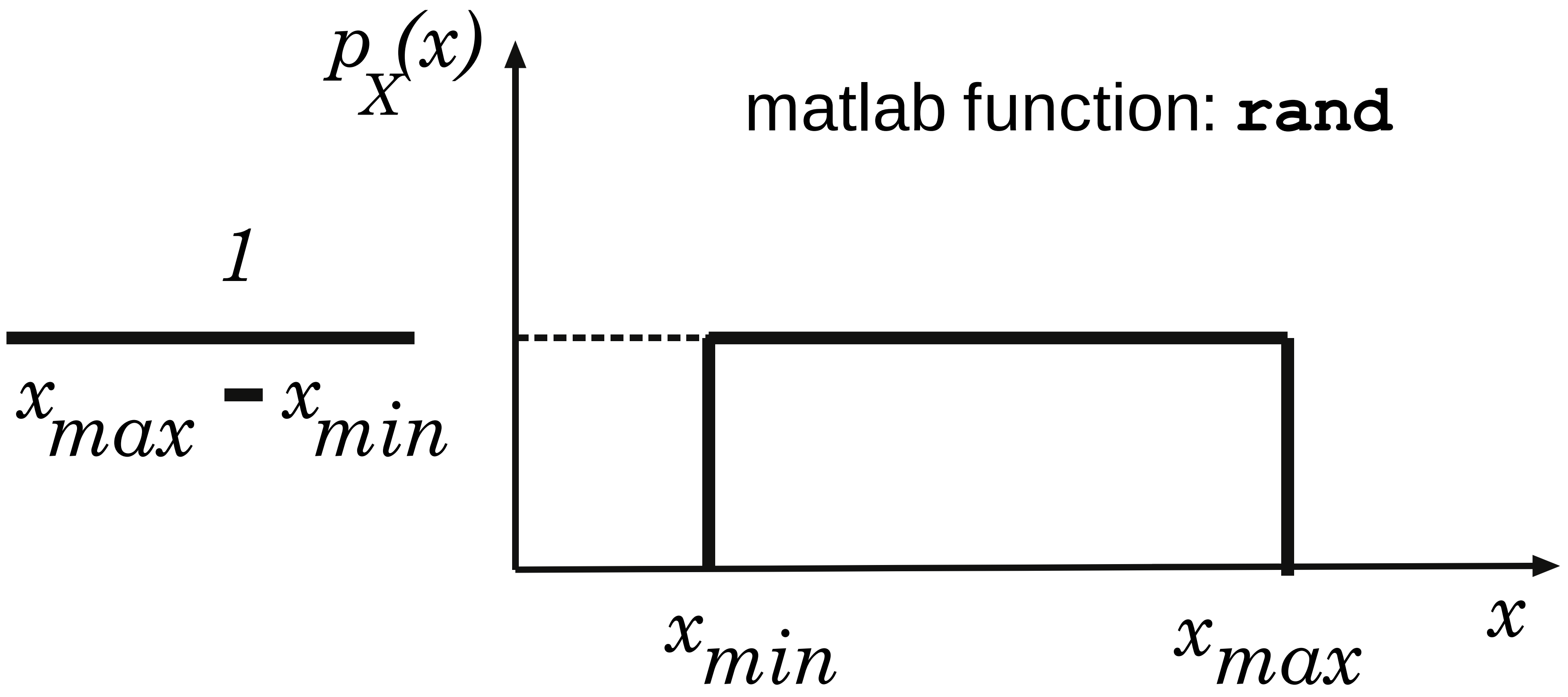
$$\left(\int_{-\infty}^{\infty} p_X(x)\, dx = 1\right)$$

$$= E[X^2] - 2\, m_X \underbrace{\int_{-\infty}^{\infty} x p_X(x)\, dx}_{m_X} + m_X^2$$

$$= E[X^2] - 2\, m_X^2 + m_X^2 \quad = \quad E[X^2] - m_X^2$$

Uniform Distribution

A random variable X which is uniformly distributed between x_{min} and x_{max} has the PDF:



matlab function: **`rand`**

Summing independent uniformly distributed random variables

- Let X and Y be 2 independent uniformly distributed variables between [0,1]

- The random variable

$$Z = X + Y$$

- is <u>**not uniformly distributed**</u>

Summing independent uniformly distributed random variables

- Let X and Y be 2 independent uniformly distributed variables between [0,1]

$$Z = X + Y$$

10^5 random samples of Z

```
X=rand(1,1e5);
Y=rand(1,1e5);
Z=X+Y;
```

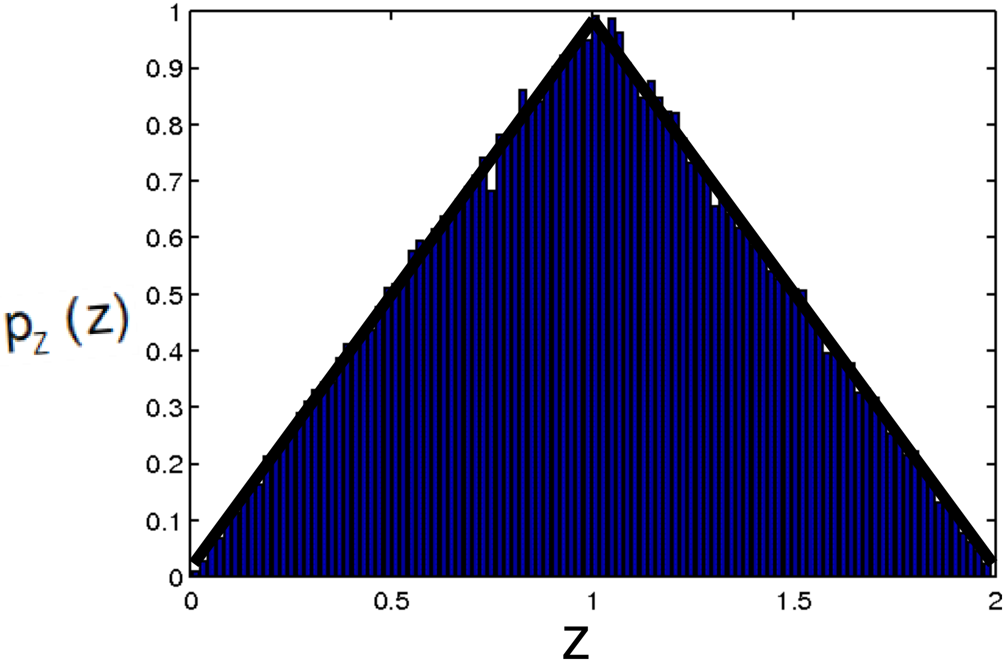
Histogram of Z with normalized area

```
[freqZ,cent]=hist(Z,100);
bin_width=(cent(100)-cent(1))/99;
area = sum(freqZ)*bin_width;
bar(centers,freqZ/area)
xlabel('z')
ylabel('F_Z(z)')
```

Summing independent uniformly distributed random variables

- Let X and Y be 2 independent uniformly distributed variables between [0,1]

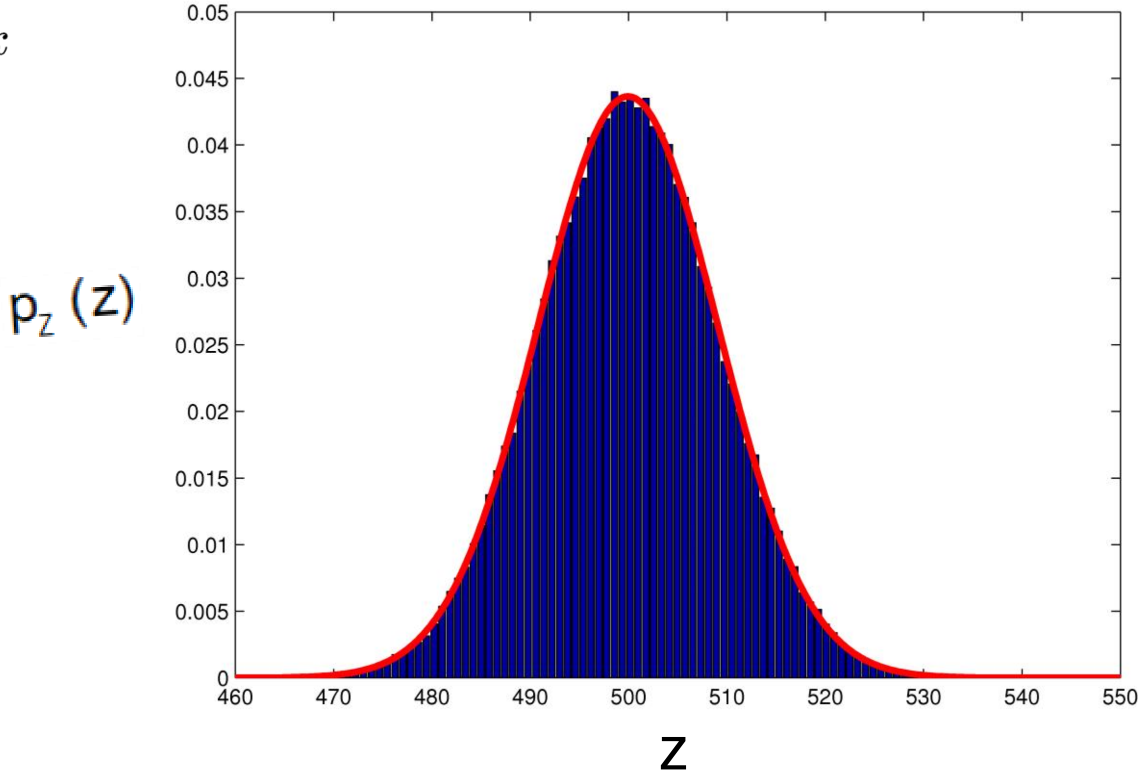
$$Z = X + Y$$



Summing a very large number of random variables

- Let $X_1,\ldots,X_{1000}$ be independent uniformly distributed variables between [0,1]

$$Z = \sum_{k=1}^{1000} X_k$$

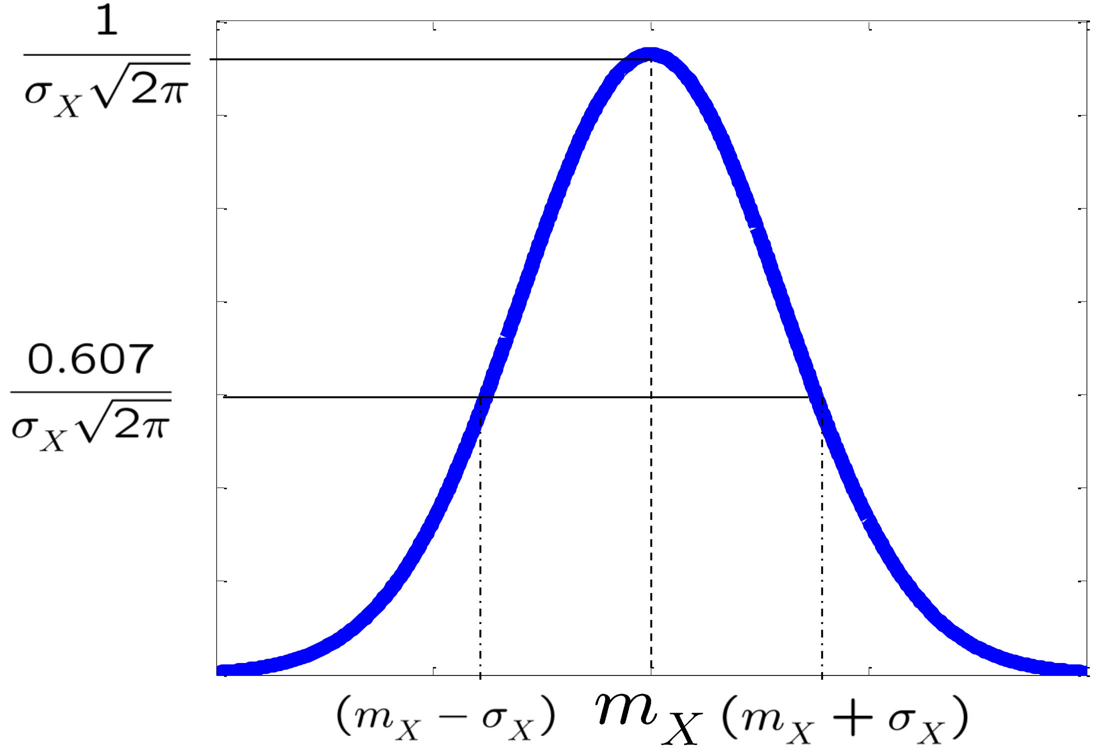


Gaussian (Normal) Distribution

$$\boxed{p_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-m_X)^2}{2\sigma_X^2}}}$$

Normal distribution

$$X \sim N(m_X, \sigma_X^2)$$



History of the Normal Distribution

From Wikipedia:

- The normal distribution was first introduced by **de Moivre** in an article in **1733** in the context of approximating certain binomial distributions for large *n*.

- His result was extended by **Laplace** in his book Analytical Theory of Probabilities (1812), and is now called the theorem of de Moivre-Laplace.

- **Laplace** used the normal distribution in the analysis of errors of experiments.

History of the Normal Distribution

From Wikipedia:

- The important method of **least squares** was introduced by **Legendre** in 1805.

- **Gauss**, who claimed to have used the method since 1794, justified it rigorously in 1809 by assuming a normal distribution of the errors.

- That the distribution is called the normal or Gaussian distribution is an instance of Stigler's law of eponymy: "No scientific discovery is named after its original discoverer."

Supplemental Material (You are not responsible for this…)

- Laplace transform of normal PDF
- Proof of the central limit theorem

Laplace transform of normal PDF

$$p_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}}\, e^{-\frac{(x-m_X)^2}{2\sigma_X^2}}$$

$$P_X(s) \;=\; \int_{-\infty}^{\infty} e^{-sx}\, p_X(x)\, dx = \frac{1}{\sigma_X\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-sx}\, e^{-\frac{(x-m_X)^2}{2\sigma_X^2}} dx$$

$$= \frac{1}{\sigma_X\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-A(x)} dx$$

where, after "completing the squares",

$$A(x) = sx + \frac{x^2}{2\sigma_X^2} + \frac{m_X^2}{2\sigma_X^2} - \frac{2m_X x}{2\sigma_X^2}$$

$$= \frac{1}{2\sigma_X^2}\left\{\left[x + (s\sigma_X^2 - m_X)\right]^2 - s^2\sigma_X^4 + 2m_X s\sigma_X^2\right\}$$

Laplace transform of normal PDF

substituting,

$$P_X(s) = e^{(s^2\sigma_X^2/2)-sm_X} \underbrace{\int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}\sigma_X} e^{-(x+s\sigma_X^2-m_X)^2/2\sigma_X^2} \right\} dx}_{=1}$$

(area under a PDF = 1)

$$\boxed{P_X(s) = e^{(s^2\sigma_X^2/2)-sm_X}}$$

Fourier transform: $P_X(j\omega) = e^{\frac{-\omega^2\sigma_X^2}{2}} e^{-j\omega m_X}$

Proof of the central limit theorem

Let X_1, X_2,... be independent random variables each with mean m_x and variance σ_x^2 and define the sequence

$$Z_n = \frac{\sum_{k=1}^{n}(X_k - m_X)}{\sqrt{n}\sigma_X} = \sum_{k=1}^{n} \frac{Y_k}{\sqrt{n}}$$

where $$Y_k = (X_k - m_X)/\sigma_X$$

notice that

$$m_Y = E\left[Y_k\right] = 0 \qquad \sigma_Y = E\left[Y_k^2\right] = 1$$

Proof of the central limit theorem

The moment generating function of Z_n is

$$P_{Z_n}(j\omega) = E\left[e^{-j\omega Z_n}\right] \quad = E\left[e^{-j\omega \sum_{k=1}^{n} \frac{Y_k}{\sqrt{n}}}\right]$$

$$= \prod_{k=1}^{n} E\left[e^{-j\omega \frac{Y_k}{\sqrt{n}}}\right]$$

by the Taylor series expansion of e^x

$$P_{Z_n}(j\omega) = \prod_{k=1}^{n} E\left[1 - \frac{j\omega Y_k}{\sqrt{n}} - \frac{\omega^2 Y_k^2}{n} - \frac{j\omega^3 Y_k^3}{n^2} + \cdots\right]$$

$$\approx \prod_{k=1}^{n}\left(1 - \frac{\omega^2}{n}\right) \;= \left(1 - \frac{\omega^2}{n}\right)^n$$

Proof of the central limit theorem

notice that, as n $\rightarrow\infty$ the approximation is exact

$$\lim_{n\rightarrow\infty} P_{Z_n}(j\omega) = \lim_{n\rightarrow\infty}\left(1-\frac{\omega^2}{n}\right)^n$$

Moreover, the PDF and moment generating function of a normally distributed random variable $X \sim N(0,1)$ are

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad P_X(j\omega) = e^{\frac{-\omega^2}{2}}$$

and $$P_X(j\omega) = e^{\frac{-\omega^2}{2}} = \lim_{n\rightarrow\infty}\left(1-\frac{\omega^2}{n}\right)^n$$

Proof of the central limit theorem

Therefore, since

$$\lim_{n\to\infty} P_{Z_n}(j\omega) = \lim_{n\to\infty}\left(1 - \frac{\omega^2}{n}\right)^n = e^{\frac{-\omega^2}{2}}$$

Then, taking the inverse Fourier transform we obtain

$$\lim_{n\to\infty} p_{Z_n}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and

$$\lim_{n\to\infty} Z_n \sim N(0,1)$$