ME 233 Advanced Control II

Lecture 19

Least Squares Parameter Estimation

Least Squares Estimation

Model

$$y(k) = \sum_{i=1}^{n} \phi_i(k-1)\,\theta_i$$

Where

- $y(k)$ observed output
- $\phi_i(k)$ **known** and measurable function
- θ_i **unknown** but constant parameter

Least Squares Estimation

Model

$$y(k) = \phi^T(k-1)\,\theta$$

Where

$y(k)$ measured output

$$\underbrace{\phi(k) = \begin{bmatrix} \phi_1(k) \\ \vdots \\ \phi_n(k) \end{bmatrix}}_{n\times 1 \text{ regressor}} \qquad \underbrace{\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}}_{\text{unknown vector}}$$

Batch Least Squares Estimation

Assume that we have collected k data sets:

$$\left.\begin{array}{l} y(1), \cdots, y(k) \\ \phi(0), \cdots, \phi(k-1) \end{array}\right\} \text{ collected data}$$

We want to find the parameter estimate at instant k: $\widehat{\theta}(k)$

that best fits **all collected** data in the **least squares** sense:

$$\min_{\widehat{\theta}(k)} \left\{ \frac{1}{2} \sum_{j=1}^{k} \left[y(j) - \phi^T(j-1)\, \widehat{\theta}(k) \right]^2 \right\}$$

kept constant in the summation

Batch Least Squares Estimation

Defining the cost functional

$$V(\widehat{\theta}(k)) = \frac{1}{2} \sum_{j=1}^{k} \left[y(j) - \phi^T(j-1)\,\widehat{\theta}(k) \right]^2$$

$\widehat{\theta}(k)$ is obtained by solving

$$\frac{dV(\widehat{\theta}(k))}{d\widehat{\theta}(k)} = 0$$

Batch Least Squares Solution

The least squares parameter estimate $\widehat{\theta}(k)$ which solves

$$\frac{dV(\widehat{\theta}(k))}{d\widehat{\theta}(k)} = 0$$

Satisfies the **normal equation:**

$$\underbrace{\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right]}_{n \times n \text{ matrix}} \widehat{\theta}(k) = \underbrace{\sum_{i=1}^{k} \phi(i-1)\, y(i)}_{n \times 1 \text{ vector}}$$

Normal Equation Derivation

$$V\left(\widehat{\theta}(k)\right) = \frac{1}{2}\sum_{j=1}^{k}\left[y(j) - \phi^T(j-1)\,\widehat{\theta}(k)\right]^2$$

$$= \frac{1}{2}\left\|\begin{bmatrix} y(1) - \phi^T(0)\widehat{\theta}(k) \\ \vdots \\ y(k) - \phi^T(k-1)\widehat{\theta}(k)\end{bmatrix}\right\|^2$$

$$= \frac{1}{2}\left\|\underbrace{\begin{bmatrix} y(1) \\ \vdots \\ y(k)\end{bmatrix}}_{Y(k)} - \underbrace{\begin{bmatrix} \phi^T(0) \\ \vdots \\ \phi^T(k-1)\end{bmatrix}}_{\Phi^T(k-1)}\widehat{\theta}(k)\right\|^2$$

Normal Equation Derivation

$$V(\hat{\theta}(k)) = \frac{1}{2}\left\|Y(k) - \Phi^T(k-1)\hat{\theta}(k)\right\|^2$$

$$= \frac{1}{2}\Big[Y^T(k)Y(k) + \hat{\theta}^T(k)\Phi(k-1)\Phi^T(k-1)\hat{\theta}(k)$$

$$-2\hat{\theta}^T(k)\Phi(k-1)Y(k)\Big]$$

Taking the partial derivative with respect to $\hat{\theta}(k)$

$$\frac{\partial V(\hat{\theta}(k))}{\hat{\theta}(k)} = \Phi(k-1)\Phi^T(k-1)\hat{\theta}(k) - \Phi(k-1)Y(k)$$

For optimality, we therefore need

$$\Phi(k-1)\Phi^T(k-1)\hat{\theta}(k) = \Phi(k-1)Y(k)$$

Normal Equation Derivation

$$\Phi(k-1) = \begin{bmatrix}\phi(0) & \cdots & \phi(k-1)\end{bmatrix}$$

$$Y(k) = \begin{bmatrix}y(1) & \cdots & y(k)\end{bmatrix}^T$$

For optimality, we need

$$\underbrace{\Phi(k-1)\Phi^T(k-1)}_{\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)}\widehat{\theta}(k) = \underbrace{\Phi(k-1)Y(k)}_{\sum_{i=1}^{k} \phi(i-1)y(i)}$$

Therefore, we need

$$\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right] \widehat{\theta}(k) = \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

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Batch Least Squares Estimation

The solution of the normal equation

$$\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right] \widehat{\theta}(k) = \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

Is given by:

$$\widehat{\theta}(k) = \left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right]^{\#} \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

$Pseudoinverse$

Moore-Penrose pseudoinverse

- Let *A* have the singular value decomposition

orthogonal matrices

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r) \qquad \sigma_1 \geq \cdots \geq \sigma_r > 0$$

- Then the Moore-Penrose pseudoinverse of *A* is

$$A^\sharp = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}$$

In MATLAB: `pinv(A)`

Moore-Penrose pseudoinverse

Let $A \in \mathcal{R}^{n \times m}$ and $A^{\sharp}$ be its Moore-Penrose pseudoinverse

Then $A^{\sharp}$ has the dimension of A^T and satisfies:

- $A\,A^{\sharp}\,A = A$
- $A^{\sharp}\,A\,A^{\sharp} = A^{\sharp}$
- $A^{\sharp}A$ and $AA^{\sharp}$ are Hermitian

In this case, since $A = \Phi\Phi^T$

$$\Phi \;=\; \left[\; \phi(0)\cdots\phi(k-1) \;\right]$$

$$\boxed{A\,A^{\sharp}\,\Phi = \Phi}$$

Batch Least Squares Estimation

Assume that we have collected sufficient data and the data has sufficient richness so that

$$\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right] = \phi(0)\phi^T(0) + \phi(1)\phi^T(1) + \cdots + \phi(k-1)\phi^T(k-1)$$

has full rank.

Then,

$$\hat{\theta}(k) = \underbrace{\left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right]^{-1}}_{F(k)} \sum_{i=1}^{k} \phi(i-1)\, y(i)$$

Recursive Least Squares (RLS)

Assume that we have collected *k-1* sets of data and have computed $\widehat{\theta}(k-1)$ using

$$\widehat{\theta}(k-1) = \underbrace{\left[\sum_{i=1}^{k-1} \phi(i-1)\phi^T(i-1)\right]^{-1}}_{F(k-1)} \sum_{i=1}^{k-1} \phi(i-1)y(i)$$

Then, given a new set of data: $y(k)$ $\phi(k-1)$

We want to find $\widehat{\theta}(k)$ in a recursive fashion:

$$\widehat{\theta}(k) = \widehat{\theta}(k-1) + [\,correction\ \ term\,]$$

Recursive Least Squares Algorithm

Define the ***a-priori*** output estimate:

$$\widehat{y}^{o}(k) = \phi^{T}(k-1)\widehat{\theta}(k-1)$$

and the ***a-priori*** output estimation error:

$$e^{o}(k) = y(k) - \phi^{T}(k-1)\widehat{\theta}(k-1)$$

The RLS algorithm is given by:

$$\boxed{\widehat{\theta}(k) \;=\; \widehat{\theta}(k-1) + F(k)\phi(k-1)\, e^{o}(k)}$$

where *F(k)* has the recursive relationship on the next slide

Recursive Least Squares Gain

The RLS gain $F(k)$ is defined by

$$F^{-1}(k) \;=\; \sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)$$

Therefore,

$$F^{-1}(k) \;=\; F^{-1}(k-1) + \phi(k-1)\phi^T(k-1)$$

Using the matrix inversion lemma, we obtain

$$\boxed{F(k) \;=\; F(k-1) - \frac{F(k-1)\,\phi(k-1)\,\phi(k-1)^T\,F(k-1)}{1+\phi(k-1)^T\,F(k-1)\,\phi(k-1)}}$$

Recursive Least Squares Derivation

Define the least squares gain matrix $F(k)$

$$\boxed{F(k) = \left[\sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)\right]^{-1}}$$

Therefore,

$$\widehat{\theta}(k) = F(k) \sum_{i=1}^{k} \phi(i-1)y(i)$$

Recursive Least Squares Derivation

Notice that

$$\widehat{\theta}(k) = F(k) \sum_{i=1}^{k} \phi(i-1)y(i)$$

$$= F(k) \left[\phi(k-1)y(k) + \underbrace{\sum_{i=1}^{k-1} \phi(i-1)y(i)}_{F^{-1}(k-1)\widehat{\theta}(k-1)} \right]$$

$$F^{-1}(k-1) \; = \; F^{-1}(k) - \phi(k-1)\phi^{T}(k-1)$$

Recursive Least Squares Derivation

Therefore plugging the previous two results,

$$\begin{aligned}\widehat{\theta}(k) &= F(k)\left[\left(F(k)^{-1} - \phi(k-1)\,\phi^T(k-1)\right)\widehat{\theta}(k-1)\right. \\ &\quad \left. + \phi(k-1)\,y(k)\right]\end{aligned}$$

And rearranging terms, we obtain

$$\begin{aligned}\widehat{\theta}(k) &= \widehat{\theta}(k-1) \\ &\quad + F(k)\phi(k-1)\,\underbrace{\left[y(k) - \phi^T(k-1)\widehat{\theta}(k-1)\right]}_{e^o(k)}\end{aligned}$$

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Recursive Least Squares Estimation

Define the ***a-priori*** output estimate:

$$\widehat{y}^{o}(k) = \phi^{T}(k-1)\widehat{\theta}(k-1)$$

and the ***a-priori*** output estimation error:

$$e^{o}(k) = y(k) - \phi^{T}(k-1)\widehat{\theta}(k-1)$$

The RLS algorithm is given by:

$$\boxed{\widehat{\theta}(k) \;=\; \widehat{\theta}(k-1) + F(k)\phi(k-1)\, e^{o}(k)}$$

Recursive Least Squares Estimation

Recursive computation of $F(k)$

$$F^{-1}(k) \;=\; \sum_{i=1}^{k} \phi(i-1)\phi^T(i-1)$$

Therefore,

$$F^{-1}(k) \;=\; F^{-1}(k-1) + \phi(k-1)\phi^T(k-1)$$

Using the matrix inversion lemma, we obtain

$$\boxed{F(k) \;=\; F(k-1) - \frac{F(k-1)\,\phi(k-1)\,\phi(k-1)^T\,F(k-1)}{1+\phi(k-1)^T\,F(k-1)\,\phi(k-1)}}$$

Recursive Least Squares Estimation

Matrix inversion lemma:

$$F^{-1}(k) \;=\; F^{-1}(k-1) + \phi(k-1)\phi^T(k-1)$$

- Multiply by $F(k-1)$ on the right and $F(k)$ on the left:

$$F(k-1) \;=\; F(k) + F(k)\,\phi(k-1)\,\phi(k-1)^T\,F(k-1)$$

- Multiply by $\phi(k-1)$ on the right:

$$\begin{aligned} F(k-1)\phi(k-1) \;&=\; F(k)\phi(k-1) \\ &\quad + F(k)\phi(k-1)\;\underbrace{\phi(k-1)^T\,F(k-1)\phi(k-1)}_{scalar} \end{aligned}$$

Recursive Least Squares Estimation

Matrix inversion lemma:

- Rearranging terms,

$$F(k-1)\phi(k-1) \;=\; \left[1+\phi(k-1)^T F(k-1)\phi(k-1)\right] F(k)\phi(k-1)$$

- Solving for $F(k)\phi(k-1)$

$$F(k)\phi(k-1) \;=\; \frac{F(k-1)\phi(k-1)}{\left[1+\phi(k-1)^T F(k-1)\phi(k-1)\right]}$$

Recursive Least Squares Estimation

Matrix inversion lemma:

- Plug

$$F(k)\phi(k-1) = \frac{F(k-1)\phi(k-1)}{\left[1+\phi(k-1)^T F(k-1)\phi(k-1)\right]}$$

into

$$F(k) = F(k-1) - F(k)\,\phi(k-1)\,\phi(k-1)^T F(k-1)$$

to obtain

$$\boxed{F(k) = F(k-1) - \frac{F(k-1)\,\phi(k-1)\,\phi(k-1)^T F(k-1)}{1+\phi(k-1)^T F(k-1)\,\phi(k-1)}}$$

RLS Estimation Algorithm

A-priori version:

$$e^{o}(k+1) = y(k+1) - \phi^{T}(k)\widehat{\theta}(k)$$

$$\widehat{\theta}(k+1) = \widehat{\theta}(k) + F(k+1)\phi(k)e^{o}(k+1)$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^{T}(k)F(k)}{1+\phi^{T}(k)F(k)\phi(k)}$$

Initial conditions:

$$F(0) = F^{T}(0) > 0 \qquad \widehat{\theta}(0)$$

RLS Estimation Algorithm

A-posteriori version (used to prove that $e(k) \longrightarrow 0$):

$$e^{o}(k+1) = y(k+1) - \phi^{T}(k)\hat{\theta}(k)$$

$$e(k+1) = \frac{e^{o}(k+1)}{1+\phi^{T}(k)F(k)\phi(k)}$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k)\phi(k)e(k+1)$$

$$F(k+1) = F(k) - \frac{F(k)\phi(k)\phi^{T}(k)F(k)}{1+\phi^{T}(k)F(k)\phi(k)}$$

RLS Estimation Algorithm

Define the parameter estimation error: $$\boxed{\tilde{\theta}(k) = \theta - \hat{\theta}(k)}$$

Notice that, since

$$y(k) = \phi^T(k-1)\theta$$

And the a-priori error is

$$e^o(k) = y(k) - \phi^T(k-1)\hat{\theta}(k-1)$$

We obtain,

$$\begin{aligned} e^o(k) &= \phi^T(k-1)\theta - \phi^T(k-1)\hat{\theta}(k-1) \\ &= \phi^T(k-1)\underbrace{\left[\theta - \hat{\theta}(k-1)\right]}_{\tilde{\theta}(k-1)} \end{aligned}$$

RLS Estimation Algorithm

Thus, the a-priori output estimation error can be written as

$$\boxed{e^o(k) \;=\; \phi^T(k-1)\tilde{\theta}(k-1)}$$

Similarly, define the **a-posteriori output and estimation error :**

$$\boxed{\begin{gathered}\widehat{y}(k) = \phi^T(k-1)\widehat{\theta}(k) \\ e(k) = y(k) - \widehat{y}(k)\end{gathered}}$$

then,

$$\boxed{e(k) \;=\; \phi^T(k-1)\tilde{\theta}(k)}$$

RLS Estimation Algorithm

Derivation of the RLS A-posteriori version:

$$\hat{\theta}(k) \;=\; \hat{\theta}(k-1) + F(k)\phi(k-1)\, e^{o}(k)$$

$$e^{o}(k) = y(k) - \phi^{T}(k-1)\hat{\theta}(k-1)$$

Remember that,

$$F(k)\phi(k-1) \;=\; \frac{F(k-1)\phi(k-1)}{\left[1+\phi(k-1)^{T}F(k-1)\phi(k-1)\right]}$$

Thus,

$$\hat{\theta}(k+1) \;=\; \hat{\theta}(k) + \frac{F(k)\phi(k)}{1+\phi^{T}(k)F(k)\phi(k)}\, e^{o}(k+1)$$

RLS Estimation Algorithm

Multiplying by $\phi^T(k)$ to the left of

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)} e^o(k+1)$$

to obtain,

$$\underbrace{\phi^T(k)\tilde{\theta}(k+1)}_{e(k+1)} = \underbrace{\phi^T(k)\tilde{\theta}(k)}_{e^o(k+1)} - \frac{\phi^T(k)F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)} e^o(k+1)$$

Thus,

$$e(k+1) = e^o(k+1) - \frac{\phi^T(k)F(k)\phi(k)}{1+\phi^T(k)F(k)\phi(k)} e^o(k+1)$$

$$= \frac{e^o(k+1)}{1+\phi^T(k)F(k)\phi(k)}$$

RLS Estimation Algorithm

Therefore, from

$$\widehat{\theta}(k+1) \;=\; \widehat{\theta}(k) + F(k)\phi(k)\frac{e^{o}(k+1)}{1+\phi^{T}(k)F(k)\phi(k)}$$

We obtain,

$$\widehat{\theta}(k+1) \;=\; \widehat{\theta}(k) + F(k)\phi(k)\, e(k+1)$$

$$\boxed{e(k+1) \;=\; \frac{e^{o}(k+1)}{1+\phi^{T}(k)F(k)\phi(k)}}$$

RLS with forgetting factor

The inverse of the gain matrix in the RLS algorithm is given by:

$$F^{-1}(k) \;=\; F^{-1}(k-1) + \phi(k-1)\phi^T(k-1)$$

Its trace is given by:

$$\mathrm{tr}\left[F^{-1}(k)\right] \;=\; \mathrm{tr}\left[F^{-1}(k-1)\right] + \|\phi(k-1)\|^2$$

which always increases when $\|\phi(k-1)\| \neq 0$

RLS with forgetting factor

Similarly, the trace of the gain matrix is given by

$$\mathrm{tr}\,[F(k)] = \mathrm{tr}\,[F(k-1)] - \frac{\|F(k-1)\phi(k-1)\|^2}{1+\phi^T(k-1)F(k-1)\phi(k-1)}$$

always decreases when $\|F(k-1)\phi(k-1)\| \neq 0$

Problem: RLS eventually stops updating

RLS with forgetting factor

We can modify cost function to “forget” old data

$$V(\widehat{\theta}(k)) = \frac{1}{2}\sum_{j=1}^{k} \lambda^{(k-j)} \left[y(j) - \phi^T(j-1)\,\widehat{\theta}(k)\right]^2$$

$$0 < \lambda \leq 1$$

Key idea: Discount old data, e.g. the term

$$\lambda^{(k-1)} \left[y(1) - \phi^T(0)\,\widehat{\theta}(k)\right]^2$$

is small when $\boldsymbol{k}$ is large since $\lim_{m\to\infty} \lambda^m = 0$

RLS with forgetting factor

A-priori version:

$$e^o(k+1) = y(k+1) - \phi^T(k)\hat{\theta}(k)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)e^o(k+1)$$

Same as RLS without forgetting factor

$$F(k+1) \;=\; \frac{1}{\lambda}\left[F(k) - \frac{F(k)\,\phi(k)\,\phi(k)^T\,F(k)}{\lambda + \phi(k)^T\,F(k)\,\phi(k)}\right]$$

$$F^{-1}(k+1) = \lambda F^{-1}(k) + \phi(k)\phi^T(k)$$

RLS with forgetting factor

A-posteriori version (used to prove that $e(k) \longrightarrow 0$):

$$e^o(k+1) = y(k+1) - \phi^T(k)\widehat{\theta}(k)$$

$$e(k+1) = \frac{\lambda e^o(k+1)}{\lambda + \phi^T(k)F(k)\phi(k)}$$

$$\widehat{\theta}(k+1) = \widehat{\theta}(k) + \frac{1}{\lambda}F(k)\phi(k)e(k+1)$$

$$F(k+1) \;=\; \frac{1}{\lambda}\left[F(k) - \frac{F(k)\,\phi(k)\,\phi(k)^T\,F(k)}{\lambda + \phi(k)^T\,F(k)\,\phi(k)}\right]$$

RLS with forgetting factor

The gain of the RLS with FF may blow up

$$\mathrm{tr}\left[F(k)\right] = \frac{1}{\lambda}\mathrm{tr}\left[F(k-1)\right] - \frac{\|F(k-1)\phi(k-1)\|^2}{\lambda^2 + \lambda\phi^T(k-1)F(k-1)\phi(k-1)}$$

if $\phi(k)$ is not persistently exciting

(more on this later)

General PAA gain formula

$$F^{-1}(k+1) = \lambda_1(k)\, F^{-1}(k) + \lambda_2(k)\, \phi(k)\phi^T(k)$$

$$0 < \lambda_1(k) \leq 1 \qquad\qquad 0 \leq \lambda_2(k) < 2$$

- Constant adaptation gain: $\lambda_1(k) = 1,\ \lambda_2(k) = 0$

  (We talked about this case in the previous lecture)

- RLS: $\lambda_1(k) = 1,\ \lambda_2(k) = 1$

- RLS with forgetting factor: $\lambda_1(k) < 1,\ \lambda_2(k) = 1$

General PAA gain formula

$$\boxed{\begin{aligned} &F^{-1}(k+1) = \lambda_1(k)\,F^{-1}(k) + \lambda_2(k)\,\phi(k)\phi^T(k) \\ &0 < \lambda_1(k) \le 1 \qquad\qquad 0 \le \lambda_2(k) < 2 \end{aligned}}$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]$$

$$F(0) = F^T(0) > 0$$

General PAA

A-priori version:

$$e^o(k+1) = y(k+1) - \phi^T(k)\widehat{\theta}(k)$$

$$\widehat{\theta}(k+1) = \widehat{\theta}(k) + \frac{1}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} F(k)\phi(k)e^o(k+1)$$

$$F(k+1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]$$

When $\lambda_2(k) = 1$, the parameter estimate equation simplifies to

$$\widehat{\theta}(k+1) = \widehat{\theta}(k) + F(k+1)\phi(k)e^o(k+1)$$

General PAA

A-posteriori version (used to prove that $e(k) \longrightarrow 0$):

$$e^o(k+1) = y(k+1) - \phi^T(k)\widehat{\theta}(k)$$

$$e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} e^o(k+1)$$

$$\widehat{\theta}(k+1) = \widehat{\theta}(k) + \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$F(k+1) = \frac{1}{\lambda_1(k)} \left[F(k) - \lambda_2(k) \frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)} \right]$$

Additional Material
(you are not responsible for this)

- The Matrix Inversion Lemma

- Relationships for the General PAA

(these will be included in the next version…)

Matrix Inversion Lemma (simplified version)

- Since $\det(I + RL) = \det(I + LR)$, we know that

$$I + RL \text{ is invertible}$$
$$\Updownarrow$$
$$I + LR \text{ is invertible}$$

- The matrix inversion lemma (simplified version) states that

$$(I + RL)^{-1} = I - R(I + LR)^{-1}L$$

Matrix Inversion Lemma (simplified version)

$$(I + RL)^{-1} = I - R(I + LR)^{-1}L$$

Proof:

Define $\Phi = I - R(I + LR)^{-1}L$

We want to show that $(I + RL)\Phi = I$

$$(I + RL)\Phi = (I + RL) - \underbrace{(I + RL)R}_{R + RLR = R(I + LR)}(I + LR)^{-1}L$$

$$(I + RL)\Phi = I + RL - R(I + LR)(I + LR)^{-1}L$$

$$= I + RL - RL$$

$\blacksquare$

Matrix Inversion Lemma

If *A*, *C*, and (*A*+*UCV*) are invertible, then

$$\boxed{(A+UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}}$$

Proof:

$$(A+UCV)^{-1} = \left[\left(I + UCVA^{-1}\right)A\right]^{-1}$$

$$= A^{-1}\left(I + UCVA^{-1}\right)^{-1}$$

$$= A^{-1}\left[I - UC\left(I + VA^{-1}UC\right)^{-1}VA^{-1}\right]$$

$$= A^{-1}\left[I - U\left[\left(I + VA^{-1}UC\right)C^{-1}\right]^{-1}VA^{-1}\right]$$

$$= A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1} \qquad \blacksquare$$

Relationships for General PAA

$$\boxed{F(k{+}1) = \frac{1}{\lambda_1(k)}\left[F(k) - \lambda_2(k)\frac{F(k)\phi(k)\phi^T(k)F(k)}{\lambda_1(k) + \lambda_2(k)\phi^T(k)F(k)\phi(k)}\right]}$$

Proof: We know that

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + [\lambda_2(k)\phi(k)]\,\phi^T(k)$$

By the Matrix Inversion Lemma

$$F(k+1) = \frac{1}{\lambda_1(k)}F(k)$$

$$-\left[\frac{1}{\lambda_1(k)}F(k)\right][\lambda_2(k)\phi(k)]\left[\frac{1}{1+\phi^T(k)\left[\frac{1}{\lambda_1(k)}F(k)\right][\lambda_2(k)\phi(k)]}\right]\phi^T(k)\left[\frac{1}{\lambda_1(k)}F(k)\right]$$

This simplifies to the stated expression for *F*(*k*+1) ∎

Relationships for General PAA

$$\boxed{F(k+1)\phi(k)=\frac{1}{\lambda_1(k)+\lambda_2(k)\phi^T(k)F(k)\phi(k)}F(k)\phi(k)}$$

Proof:

$$F^{-1}(k+1)=\lambda_1(k)F^{-1}(k)+\lambda_2(k)\phi(k)\phi^T(k)$$

$$\Downarrow$$

$$\begin{aligned}F(k+1)\left[F^{-1}(k+1)\right]F(k)\phi(k)\\ =F(k+1)\left[\lambda_1(k)F^{-1}(k)+\lambda_2(k)\phi(k)\phi^T(k)\right]F(k)\phi(k)\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}F(k)\phi(k)&=\lambda_1(k)F(k+1)\phi(k)\\ &\quad+\lambda_2(k)F(k+1)\phi(k)\phi^T(k)F(k)\phi(k)\\ &=F(k+1)\phi(k)\left[\lambda_1(k)+\lambda_2(k)\phi^T(k)F(k)\phi(k)\right]\end{aligned}$$

$\blacksquare$

Relationships for General PAA

$$\boxed{e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} e^o(k+1)}$$

Proof:

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)$$

$$\Downarrow$$

$$\underbrace{\phi^T(k)\tilde{\theta}(k+1)}_{e(k+1)} = \phi^T(k)\left[\tilde{\theta}(k) - \frac{1}{\lambda_1(k)} F(k)\phi(k)e(k+1)\right]$$

$$= \underbrace{\phi^T(k)\tilde{\theta}(k)}_{e^o(k+1)} - \frac{1}{\lambda_1(k)} \phi^T(k)F(k)\phi(k)e(k+1)$$

Relationships for General PAA

$$\boxed{e(k+1) = \frac{\lambda_1(k)}{\lambda_1(k) + \phi^T(k)F(k)\phi(k)} e^o(k+1)}$$

Proof (continued):

From the previous slide,

$$e(k+1) = e^o(k+1) - \frac{1}{\lambda_1(k)}\phi^T(k)F(k)\phi(k)e(k+1)$$

$$\Downarrow$$

$$\left[\lambda_1(k) + \phi^T(k)F(k)\phi(k)\right] e(k+1) = \lambda_1(k)e^o(k+1)$$

■