ME 233 Advanced Control II

Lecture 18

Stability Analysis Using The Hyperstability Theorem

Adaptive Control

Basic Adaptive Control Principle

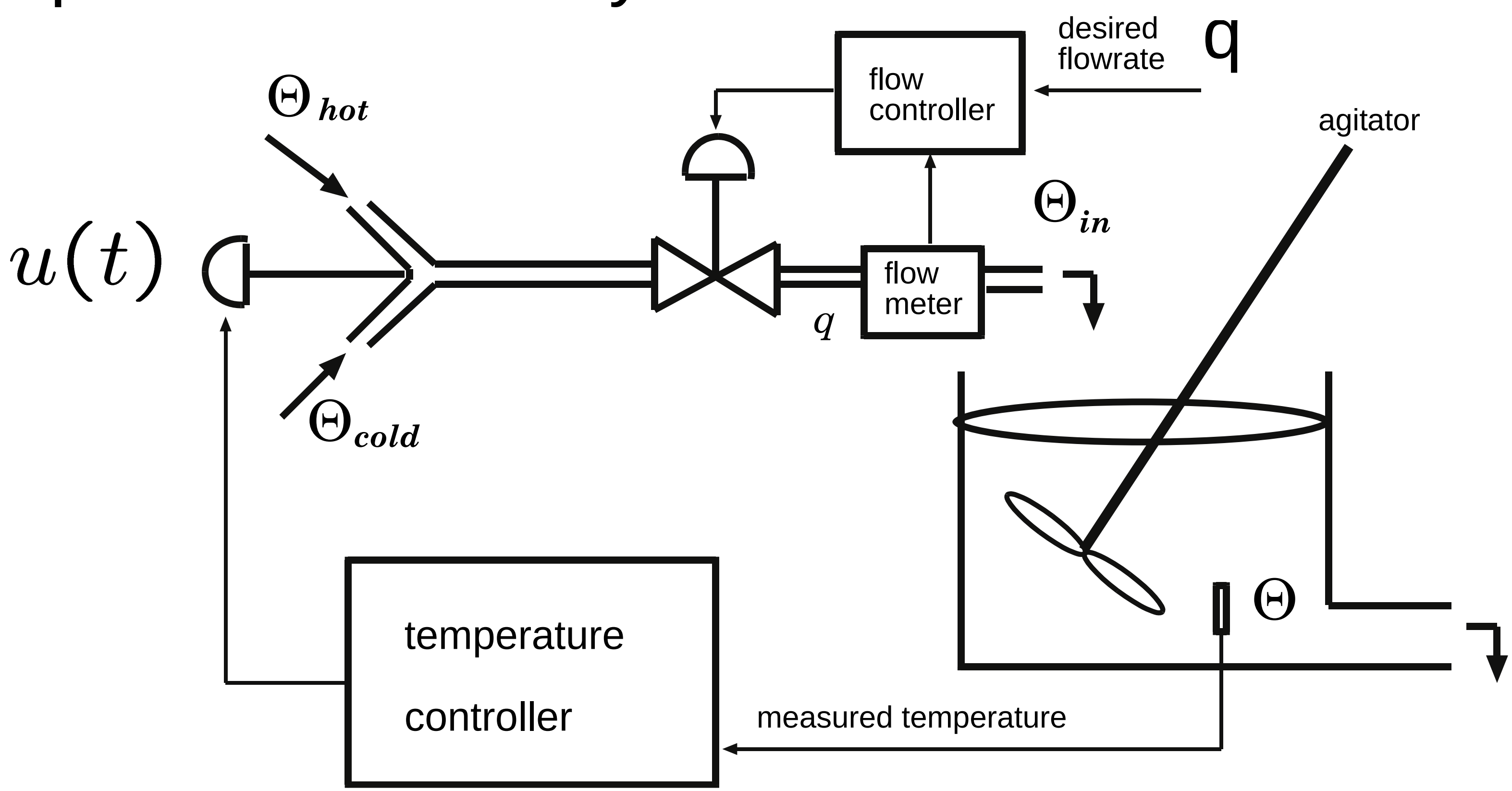
Controller parameters **are not constant,** rather, they are adjusted in an online fashion by a ***Parameter Adaptation Algorithm (PAA)***

When is adaptive control used?

- Plant parameters are unknown
- Plant parameters are time varying

Example of a system with varying parameters

- Temperature control system



$$\frac{d}{dt}\theta(t) = -\underbrace{\frac{q}{V}}_{a(q)}\theta(t) + \underbrace{\frac{kq}{V}}_{b(q)}u(t-t_d)$$

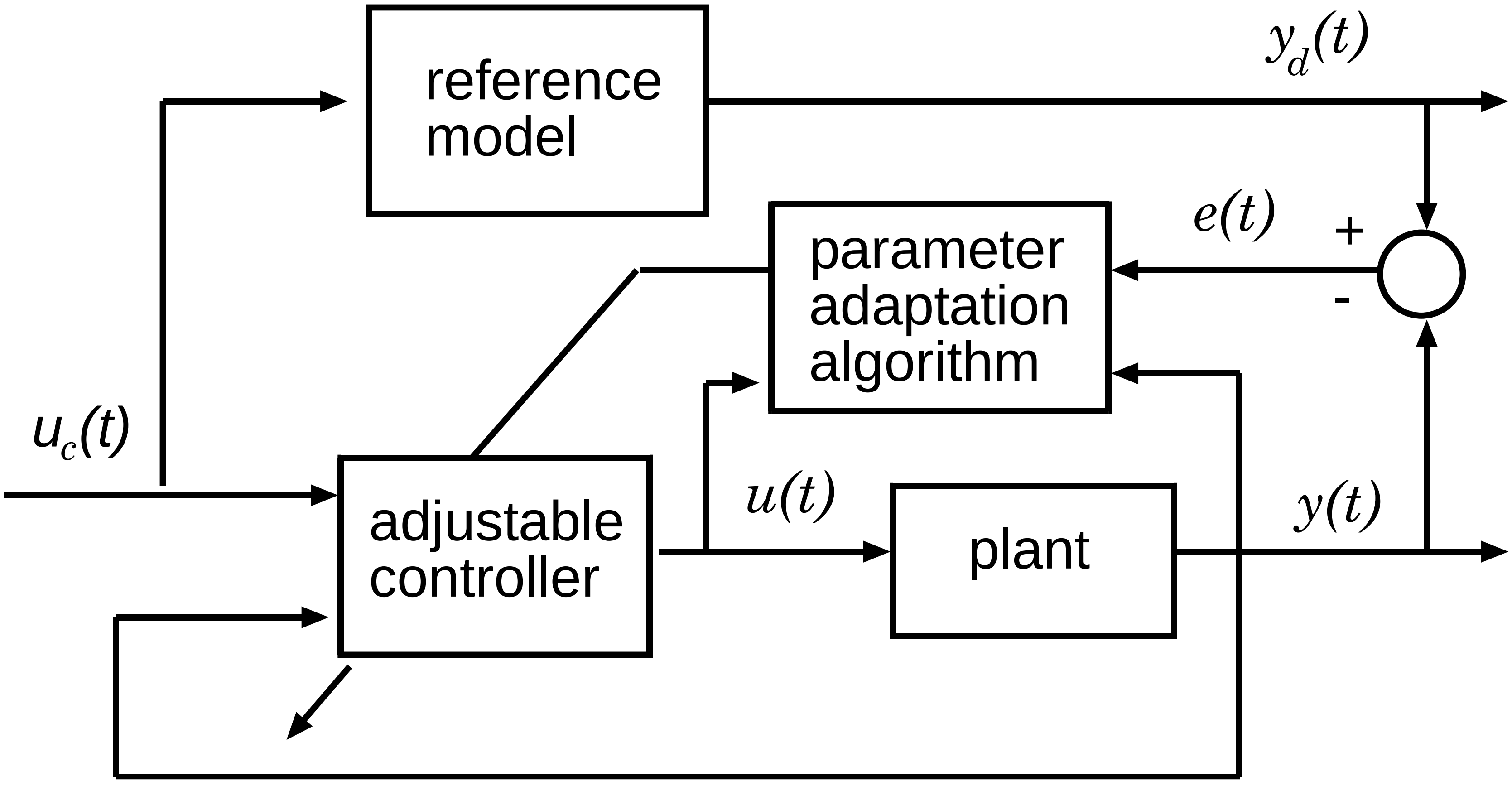
Adaptive Control Classification

- Continuous time VS **discrete time**

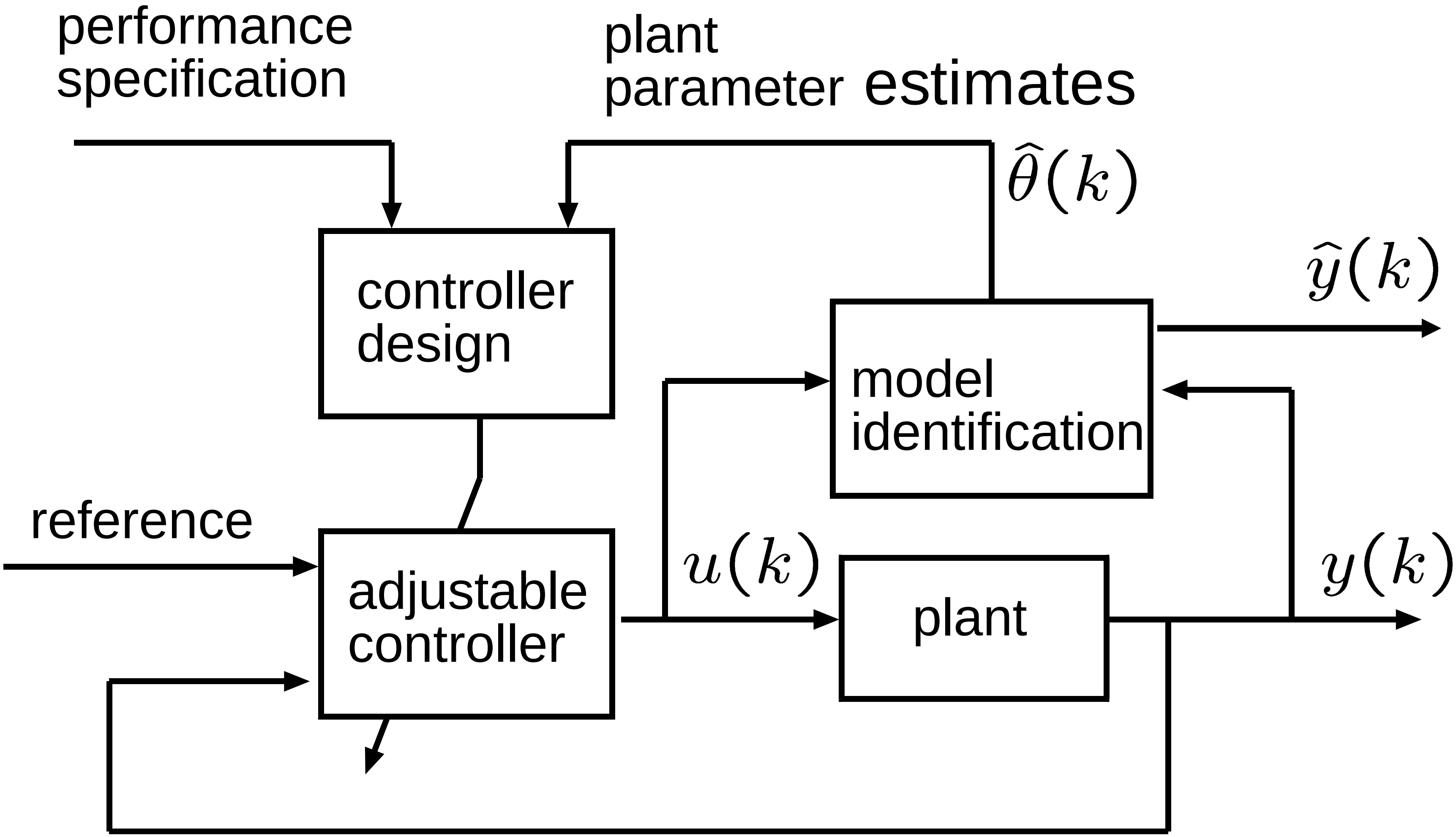
- Direct VS indirect

- MRAS VS **STR**

Model Reference Adaptive Systems (MRAS)

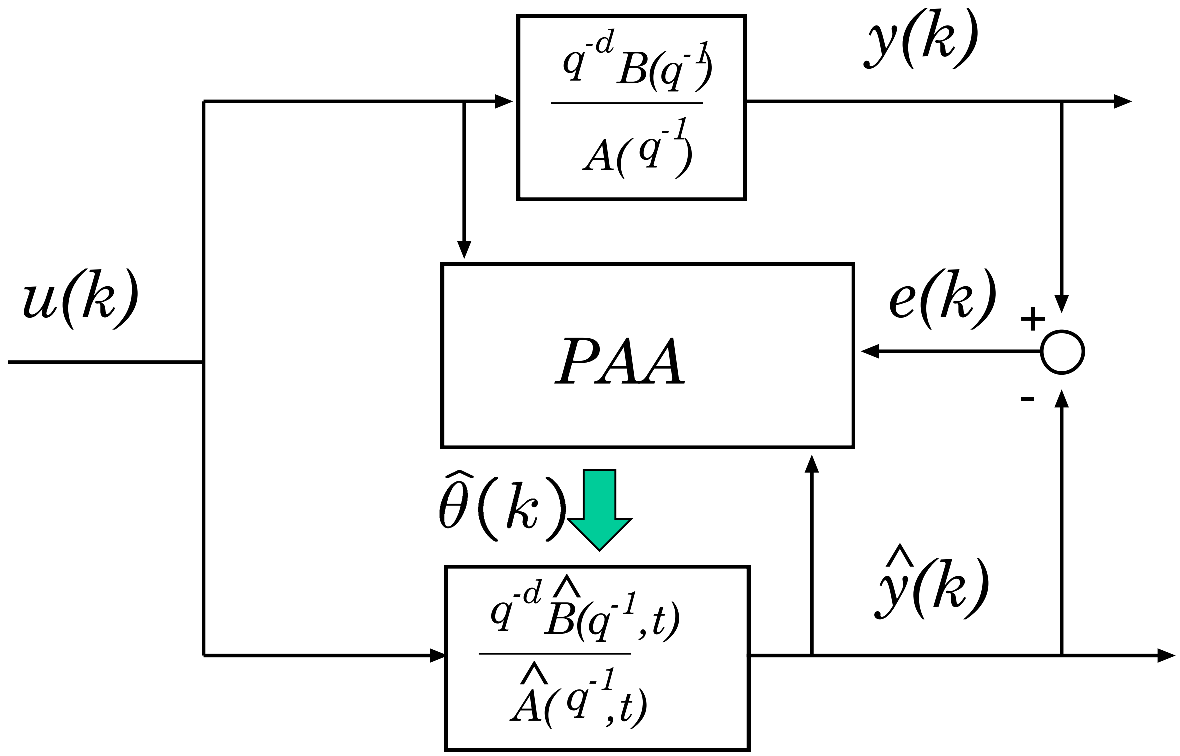


Self-Tuning Regulators (STR)



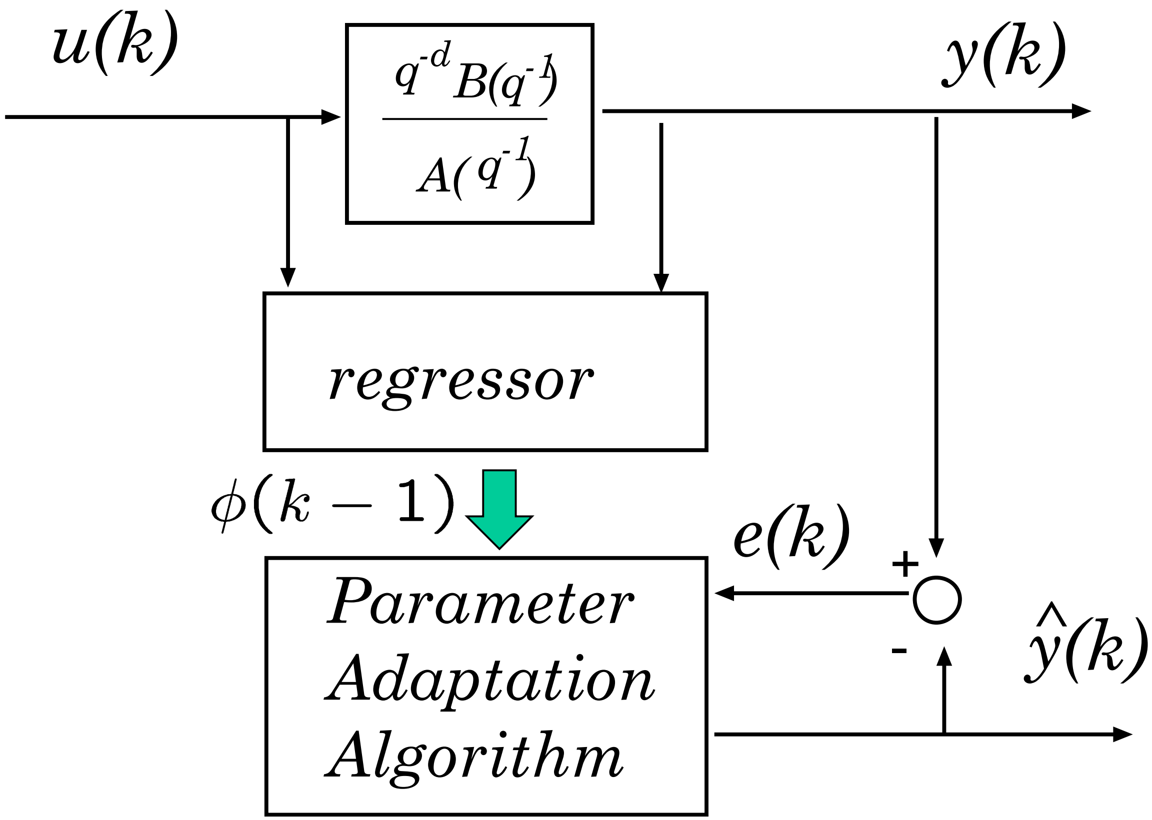
$$e(k) = y(k) - \widehat{y}(k)$$

Identification of a LTI system



Parallel model

Identification of a LTI system



Series-parallel model

(We will use this model throughout this lecture)

Plant ARMA Model

Plant model

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k)$$

where

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} \\ B(q^{-1}) &= b_o + b_1 q^{-1} + \cdots + b_m q^{-m} \end{aligned}$$

Unknown plant parameters

Assume ARMA model parameters are **unknown**

$$y(k) = -\underline{a_1}\, y(k-1) \cdots - \underline{a_n}\, y(k-n)$$

$$+\, \underline{b_o} u(k-d) \cdots + \underline{b_m}\, u(k-d-m)$$

Define:

$$\theta = \begin{bmatrix} \underline{a_1 \quad \cdots \quad a_n} & \underline{b_o \quad \cdots \quad b_m} \end{bmatrix}^T$$

As the *unknown* parameter vector

Regressor vector

Collect all measurable signals in one vector

$$y(k) = -\ a_1\, \underline{y(k-1)} \cdots -\ a_n\, \underline{y(k-n)}$$

$$+\ b_o \underline{u(k-d)} \cdots +\ b_m\, \underline{u(k-d-m)}$$

We define

$$\phi(k-1) = [\underline{-y(k-1)\ \cdots\ -y(k-n)}$$
$$\underline{u(k-d)\ \cdots\ u(k-d-m)}]^T$$

as the ***known*** regressor vector

Plant ARMA Model

Plant model

$$y(k) = \phi^T(k-1)\,\theta$$

where

$$\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_o & \cdots & b_m \end{bmatrix}^T$$

$$\phi(k-1) = [-y(k-1) \ \cdots \ -y(k-n)$$
$$u(k-d) \ \cdots \ u(k-d-m)]^T$$

Plant ARMA Model

Plant estimate (series-parallel)

$$\widehat{y}(k) = \phi^T(k-1)\,\widehat{\theta}(k)$$

where

$$\widehat{\theta}(k) = \left[\begin{array}{cccccc} \widehat{a}_1(k) & \cdots & \widehat{a}_n(k) & \widehat{b}_o(k) & \cdots & \widehat{b}_m(k) \end{array}\right]^T$$

$$\begin{aligned}\phi(k-1) = [-y(k-1) \;\; \cdots \;\; -y(k-n) \\ u(k-d) \; \cdots \; u(k-d-m)]^T\end{aligned}$$

Plant output estimate

Plant a-posteriori estimate

$$\widehat{y}(k) = \phi^T(k-1)\,\widehat{\theta}(k)$$

Plant a-priori estimate

$$\widehat{y}^o(k) = \phi^T(k-1)\,\widehat{\theta}(k-1)$$

Plant a-posteriori error

$$y(k) = \phi^T(k-1)\,\theta$$

$$\widehat{y}(k) = \phi^T(k-1)\,\widehat{\theta}(k)$$

error: $$e(k) = y(k) - \widehat{y}(k)$$

$$e(k) = \phi^T(k-1)\left[\theta - \widehat{\theta}(k)\right]$$

$$= \phi^T(k-1)\tilde{\theta}(k)$$

$$\tilde{\theta}(k) = \theta - \widehat{\theta}(k)$$

A Parameter Adaptation Algorithm

PAA $F = F^T \succ 0$

$$\hat{\theta}(k) = \hat{\theta}(k-1) + F\,\phi(k-1)e(k)$$

Parameter error update law: $\tilde{\theta}(k) = \theta - \hat{\theta}(k)$

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Adaptation Dynamics

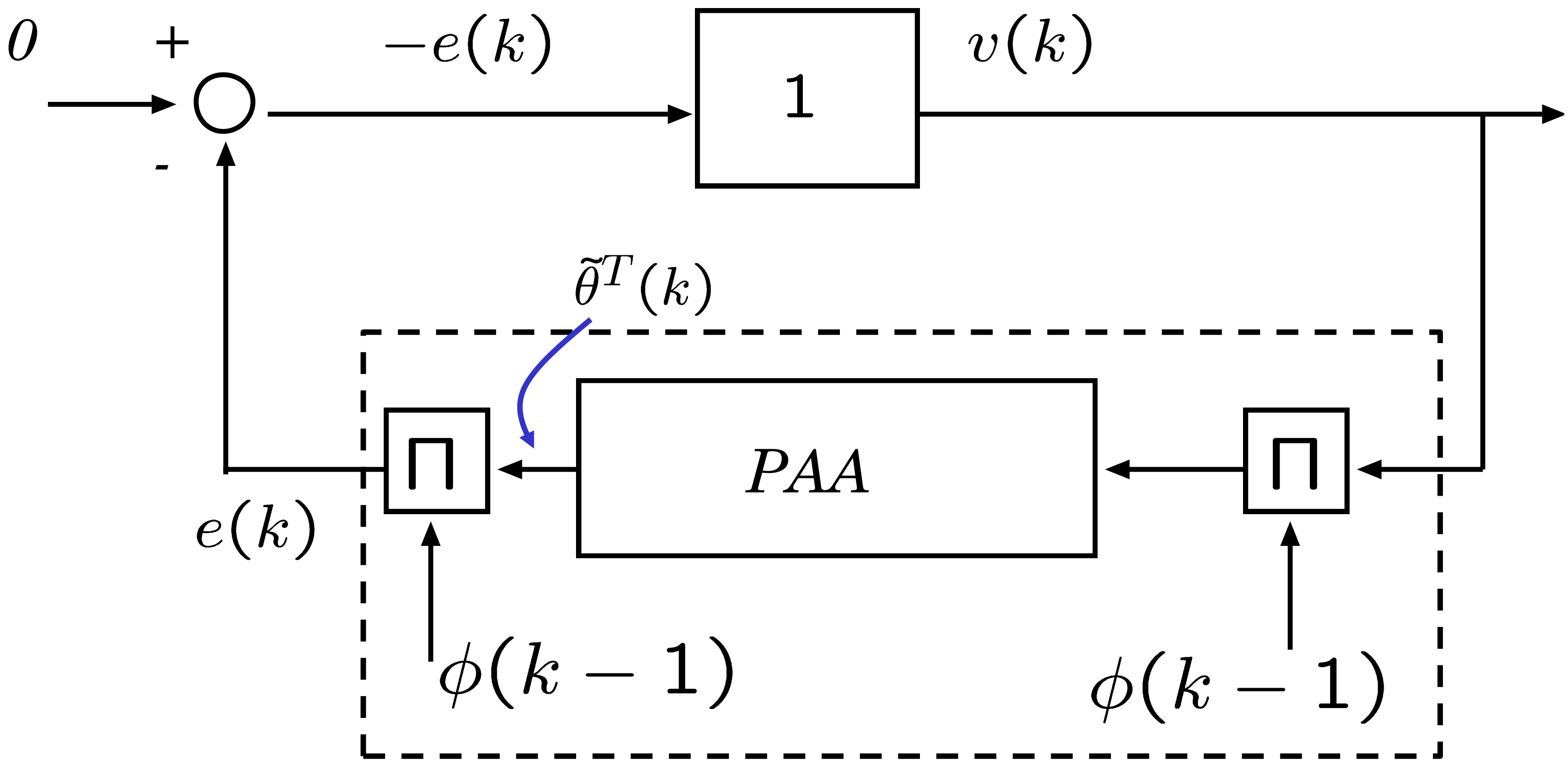
a-posteriori error: $$e(k) = y(k) - \widehat{y}(k)$$

$$e(k) = \phi^T(k-1)\tilde{\theta}(k)$$

Parameter error update law: $$\tilde{\theta}(k) = \theta - \widehat{\theta}(k)$$

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Adaptation Dynamics



PAA: $$\tilde{\theta}(k) = \tilde{\theta}(k-1) + F\,\phi(k-1)v(k)$$

Convergence of Adaptive Systems

Adaptive systems are nonlinear

We need to prove that the algorithms converge:

- **Output error convergence**

$$\boxed{e(k) \to 0}$$

$$e(k) = y(k) - \widehat{y}(k)$$

- **Parameter error convergence**

$$\boxed{\widetilde{\theta}(k) \to 0}$$

$$\widetilde{\theta}(k) = \theta - \widehat{\theta}(k)$$

Output error Convergence

Our first goal will be to prove the asymptotic convergence of the output error:

$$e(k) \longrightarrow 0$$

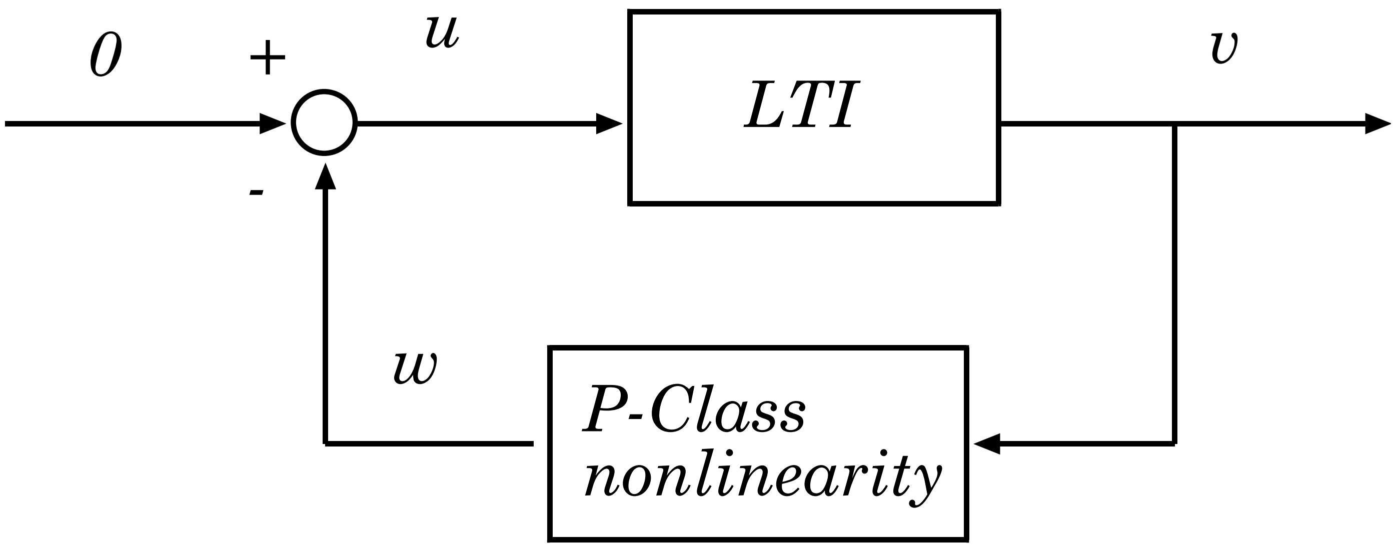
Two frequently used methods of stability analysis are**:**

- **Stability analysis using Lyapunov's direct method**
  - State space approach
- **Stability analysis using the Passivity or Hyperstability theorems**
  - Input/output approach

Hyperstability

Hyperstability Theory

- Developed by V.M. Popov to analyze the stability of a class of feedback systems (monograph published in 1973)

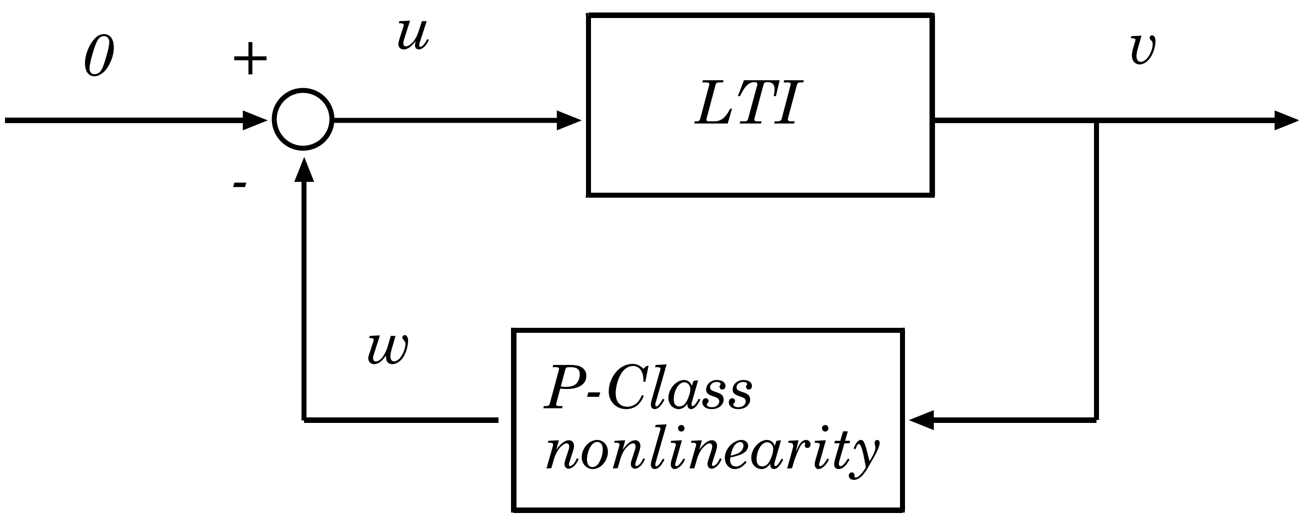


- Popularized by I.D. Landau for the analysis of adaptive systems (first book published in 1979)

Hyperstability Theory

Hyperstability Theory

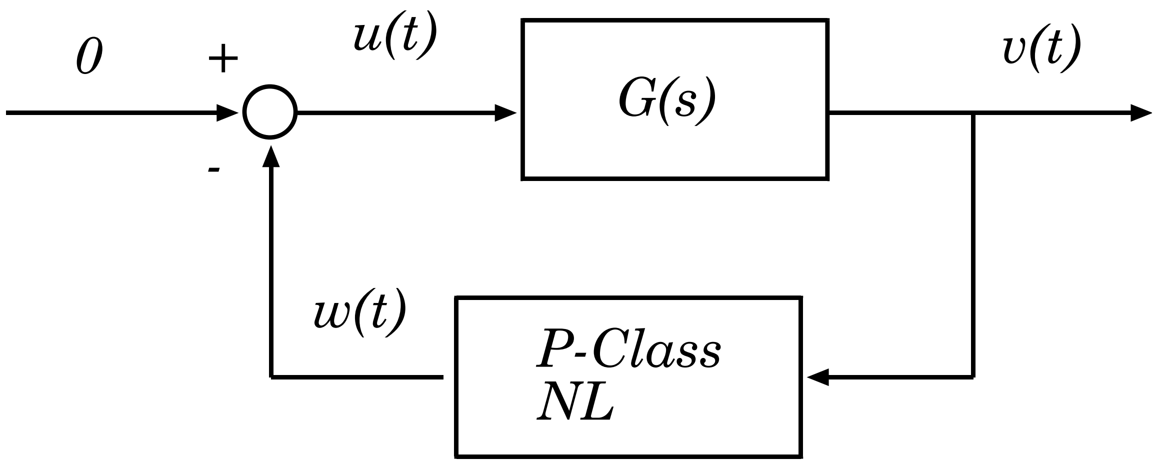
- Applies to both continuous time and discrete time systems



- **Abuse of notation:** We will denote the LTI block by its transfer function

CT Hyperstability Theory

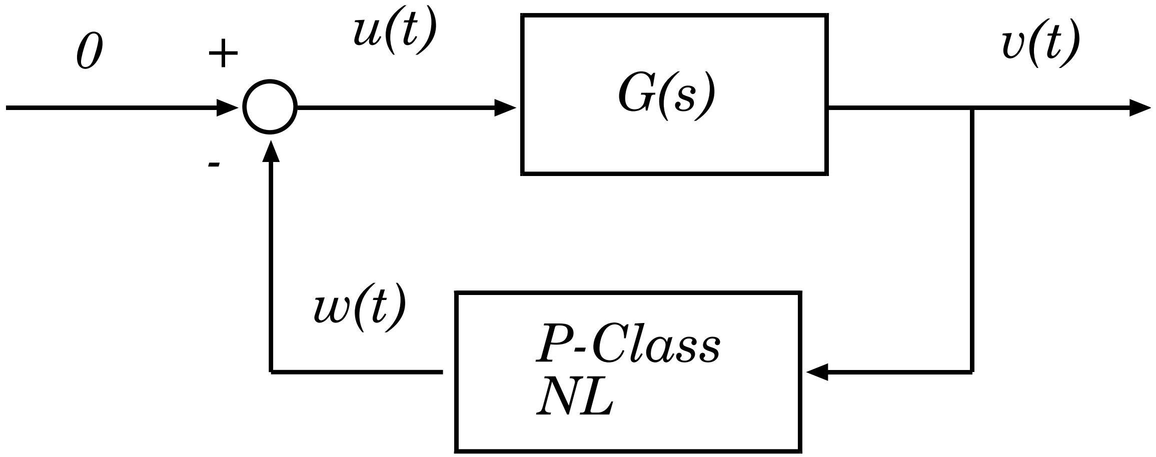
$$G(s) = C(sI - A)^{-1}B + D$$



- A state space description of the LTI Block:

$$\begin{aligned} \frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) \\ v(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

CT Hyperstability Theory

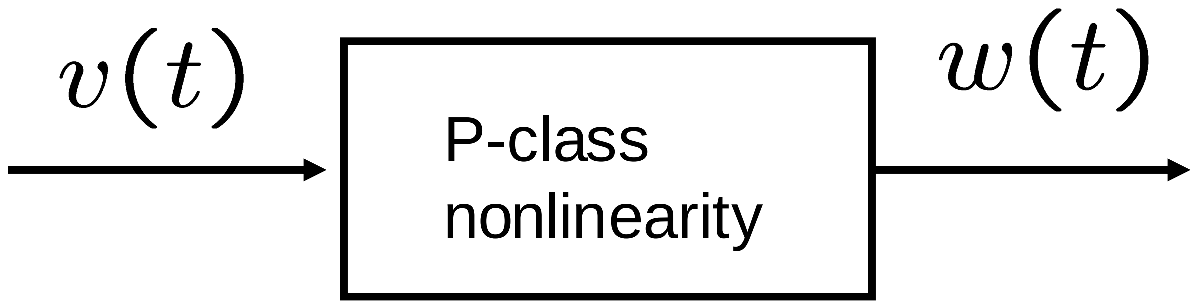


- P-class nonlinearity: (passive nonlinearities)

$$\boxed{\int_0^t w^T v \, d\tau \geq -\gamma_o^2 \qquad \forall \, t \geq 0}$$

Where γ_o is a constant which is a function of the initial conditions

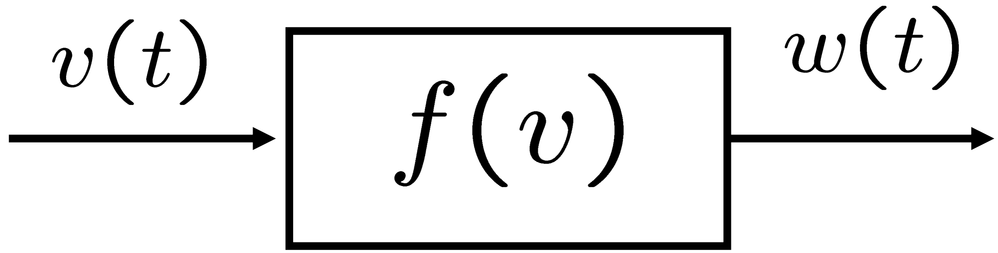
CT Hyperstability Theory



$$\boxed{\int_0^t w^T v\, d\tau \geq -\gamma_o^2 \qquad\qquad \forall\, t \geq 0}$$

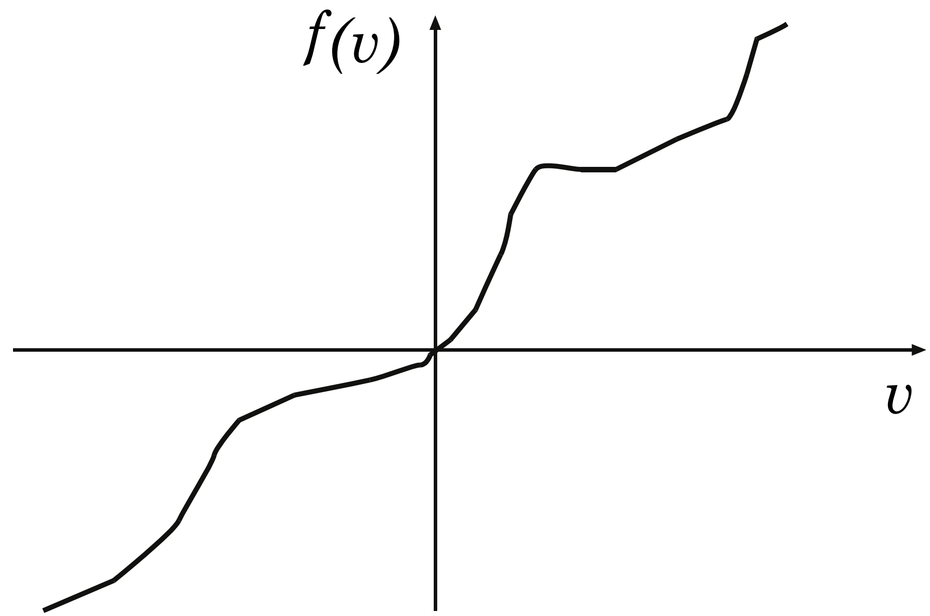
Where γ_o is a constant which is a function of the initial conditions

Example: Static P-class NL



$$w = f(v)$$

$$v\, f(v) \geq 0$$



Example: Static P-class NL

$$w = f(v)$$

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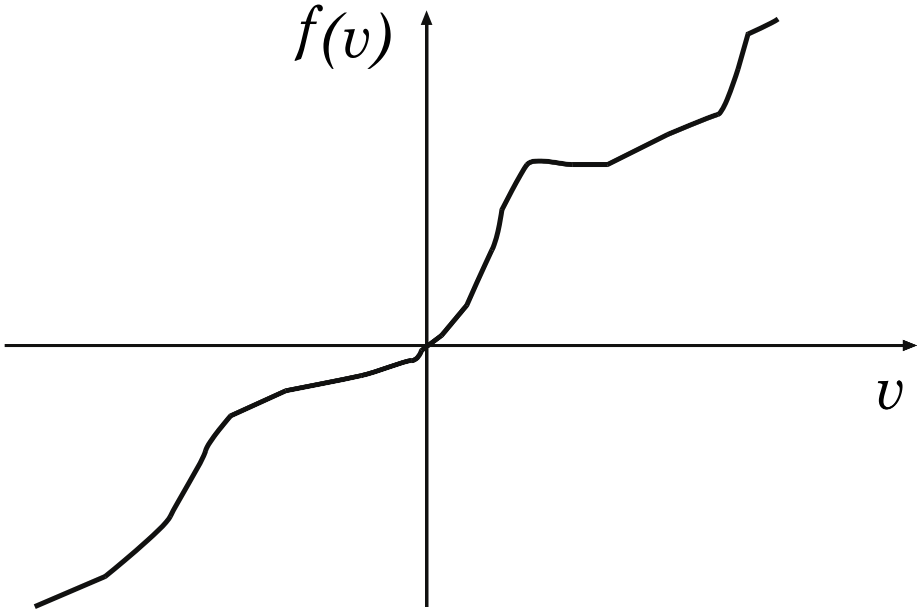


$$\int_0^t wv\,d\tau = \int_0^t \underbrace{f(v)v\,d\tau}_{\geq 0} \geq 0 > -\gamma_o^2$$

Example: Dynamic P-class block

$$\xrightarrow{v(t)} \boxed{\text{P-NL}} \xrightarrow{w(t)}$$

$$\begin{cases} \dfrac{d}{dt}\tilde{\theta}(t) = F\,\phi(t)v(t) \\ w(t) = \phi^T(t)\tilde{\theta}(t) \end{cases}$$

$$\phi(t) \in \mathcal{R}^n$$

$$\tilde{\theta}(0) \in \mathcal{R}^n$$

$$|\tilde{\theta}(0)| < \infty$$

$$|\phi(t)| < \infty$$

$$F = F^T \succ 0$$

Example: Dynamic P-class block

$$w(t) = \phi^T(t)\tilde{\theta}(t) \qquad \dot{\tilde{\theta}}(t) = F\,\phi(t)v(t)$$

$$\int_0^t w(\tau)v(\tau)\,d\tau = \int_0^t \phi^T(\tau)\tilde{\theta}(\tau)v(\tau)d\tau$$

$$= \int_0^t \tilde{\theta}^T(\tau)\underbrace{[\phi(\tau)\tilde{v}(\tau)]}_{F^{-1}\dot{\tilde{\theta}}(\tau)}d\tau$$

$$= \frac{1}{2}\int_0^t \frac{d}{d\tau}\left\{\tilde{\theta}^T(\tau)F^{-1}\tilde{\theta}(\tau)\right\}d\tau$$

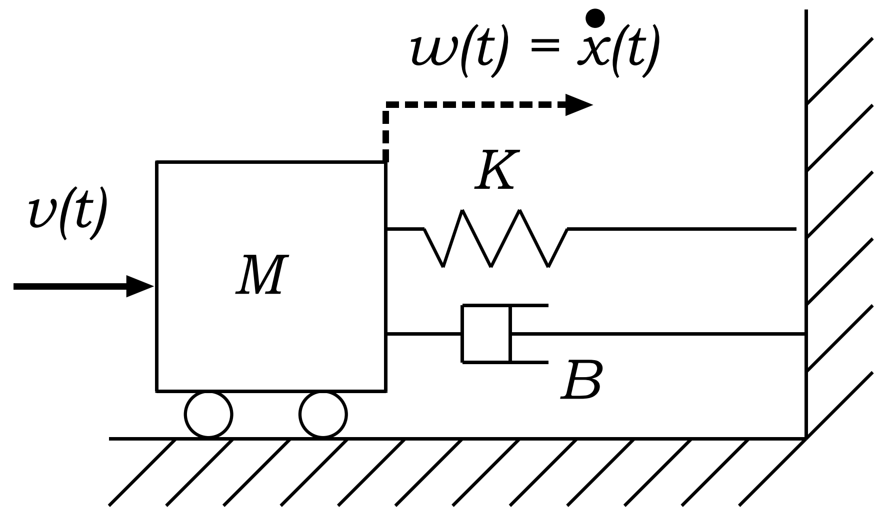
$$= \frac{1}{2}\tilde{\theta}^T(t)F^{-1}\tilde{\theta}(t) - \underbrace{\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)}_{\gamma_o^2}$$

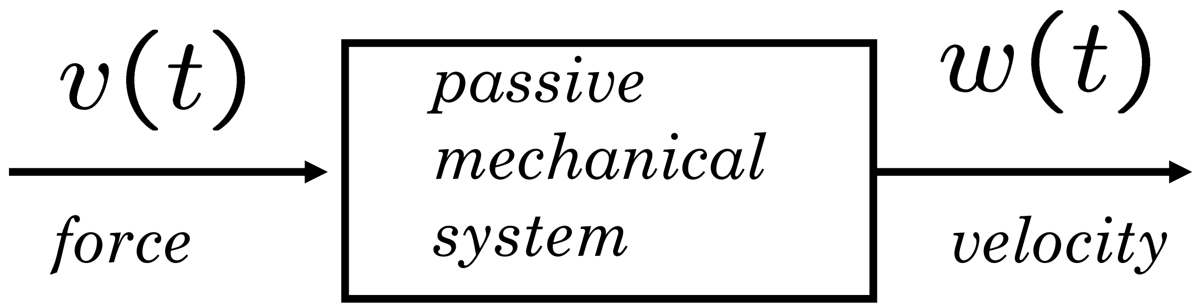
$$\geq -\gamma_o^2$$

Example: Passive mechanical system

Input is force and output is velocity

$$M\dot{w} + Bw + Kx = v$$

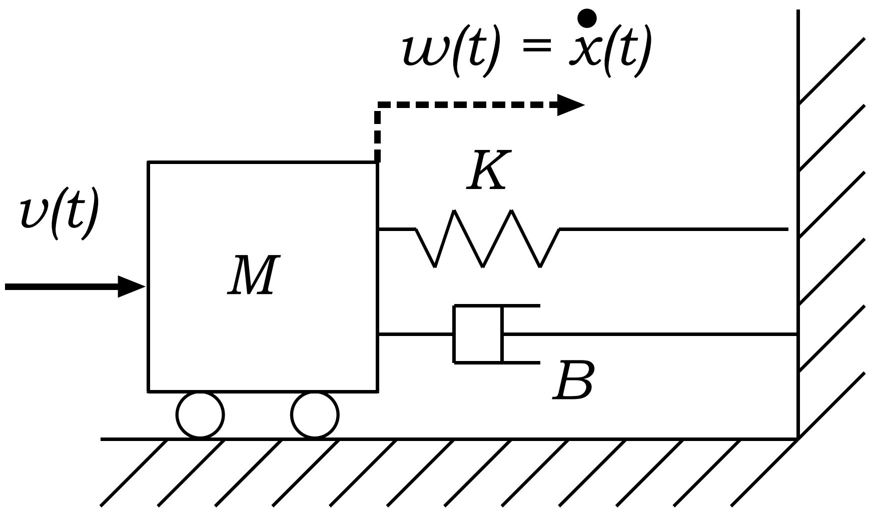




Example: Passive mechanical system

Input is force and output is velocity

$$M\dot{w} + Bw + Kx = v$$
$$\dot{x} = w$$

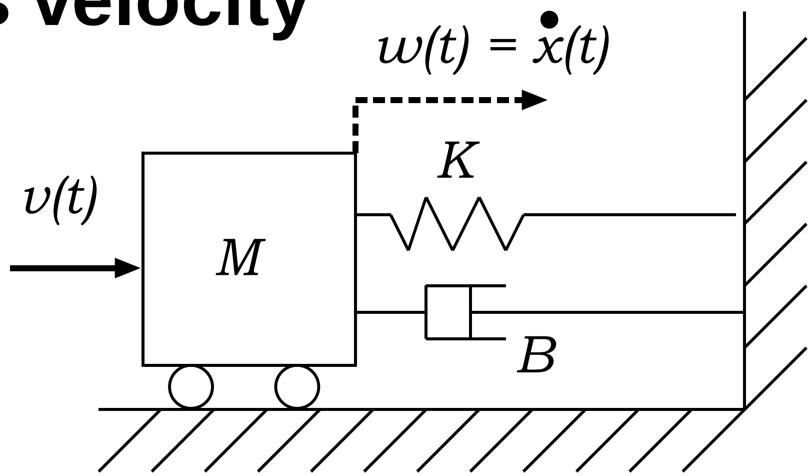


System Energy:

$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$

Example: Passive mechanical system

Input is force and output is velocity



$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$

Differentiating energy

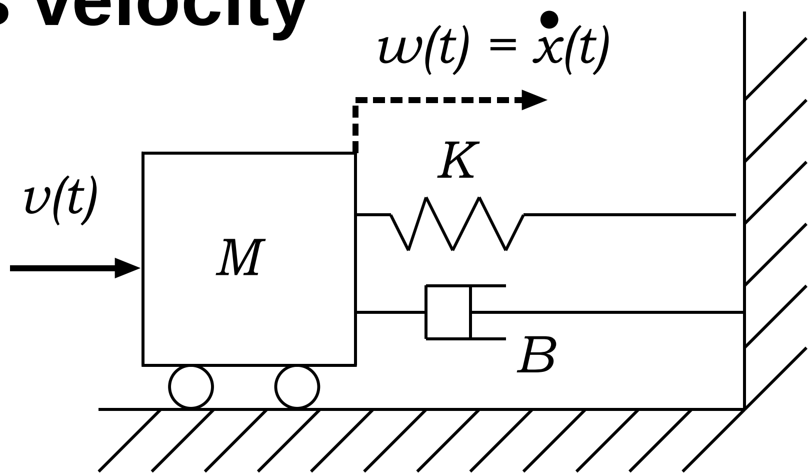
$$\dot{E} = M\dot{w}w + Kxw$$

$$= [-Kx - Bw + v]\,w + Kxw$$

$$= \cancel{-Kxw} - Bw^2 + wv + \cancel{Kxw}$$

Example: Passive mechanical system

Input is force and output is velocity



$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$

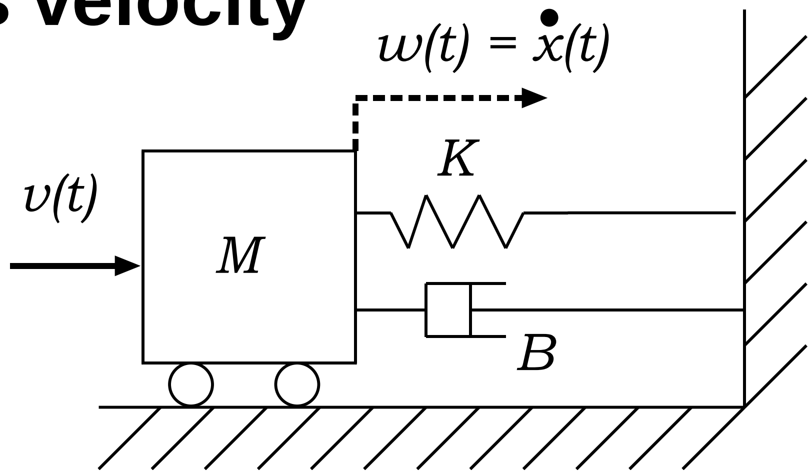
Differentiating energy

$$\dot{E} = -Bw^2 + w\,v$$

$$\underbrace{w\,v}_{power\ input} = \dot{E} + Bw^2$$

Example: Passive mechanical system

Input is force and output is velocity



$$E(t) = \frac{1}{2}Mw^2(t) + \frac{1}{2}Kx^2(t) \geq 0$$

integrating power,

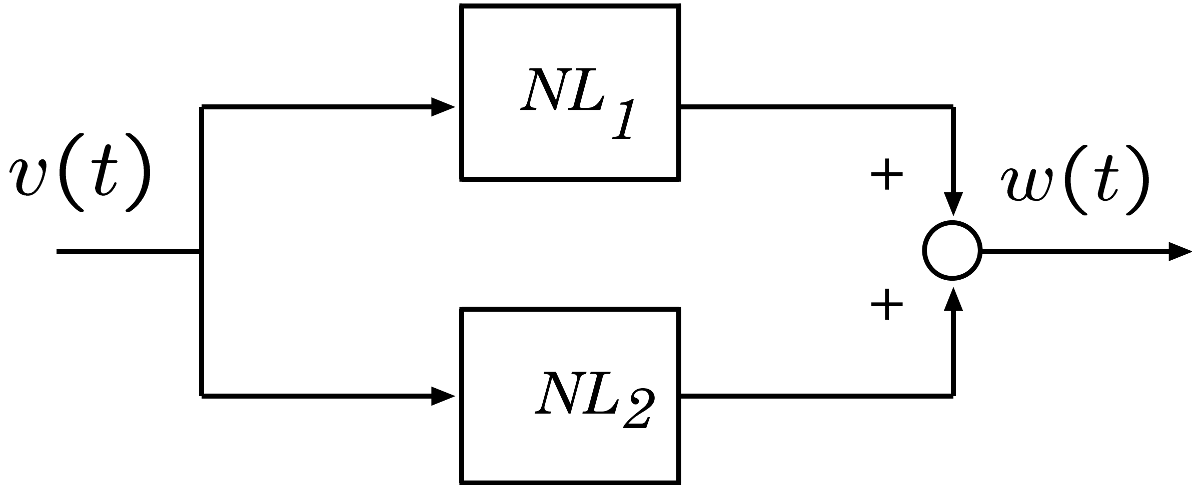
$$\begin{aligned} \int_0^t wv\,d\tau &= E(t) - E(0) + \int_0^t Bw^2(\tau)\,d\tau \\ &\geq -\gamma_o^2 \end{aligned}$$

$$\gamma_o^2 = E(0) \geq 0$$

Examples of P-class NL

Lemma:

- The parallel combination of two P-class nonlinearities is also a P-class nonlinearity.

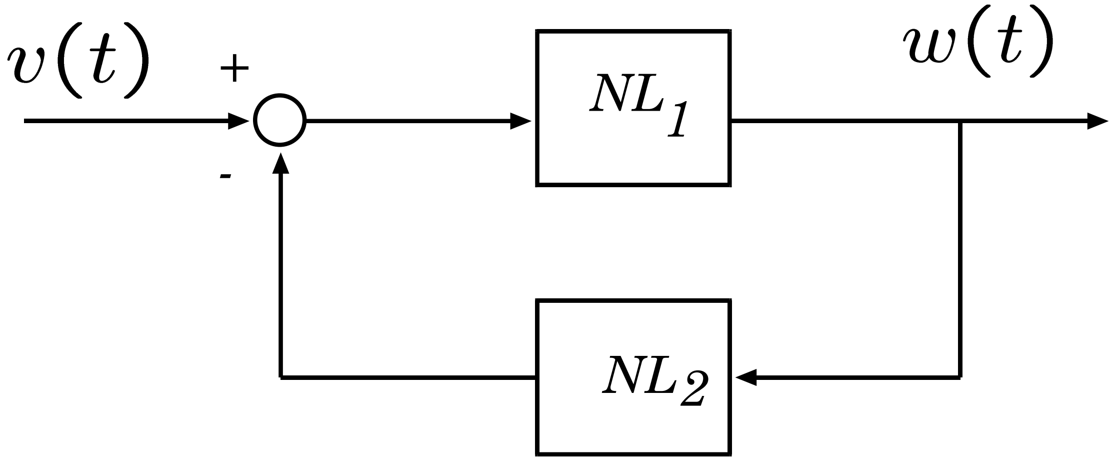


$$\int_0^t w^T v \, d\tau \geq -\gamma_o^2$$

Examples of P-class NL

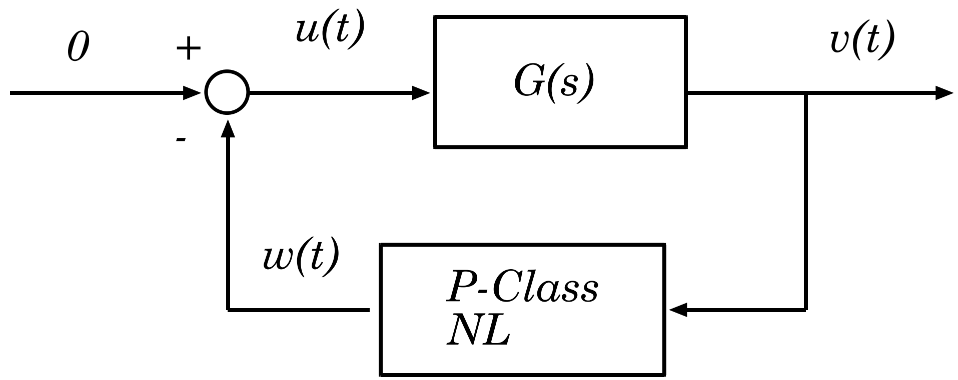
Lemma:

- The feedback combination of two P-class nonlinearities is also a P-class nonlinearity.



$$\int_0^t w^T v \, d\tau \geq -\gamma_o^2$$

CT Hyperstability

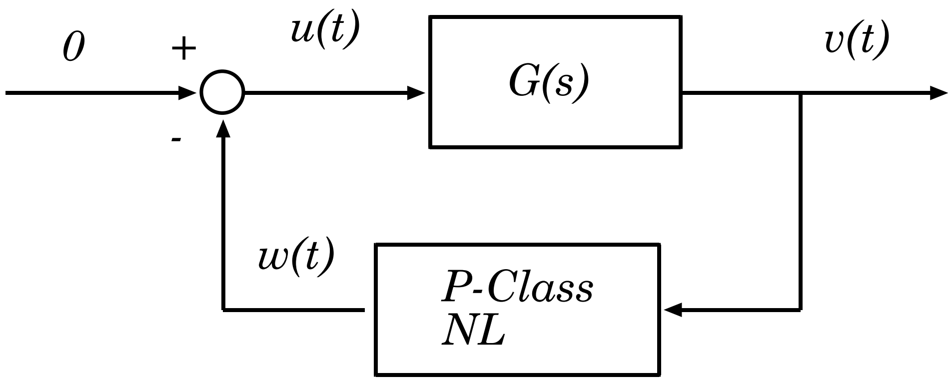


Hyperstability: The above feedback system is hyperstable if there exist positive bounded constants δ_1, δ_2 such that, for any state space realization of *G(s)*,

$$|x(t)| < \delta_1 \left[|x(0)| + \delta_2 \right] \qquad \forall\, t \geq 0$$

FOR ALL P-class nonlinearities

CT Asymptotic Hyperstability



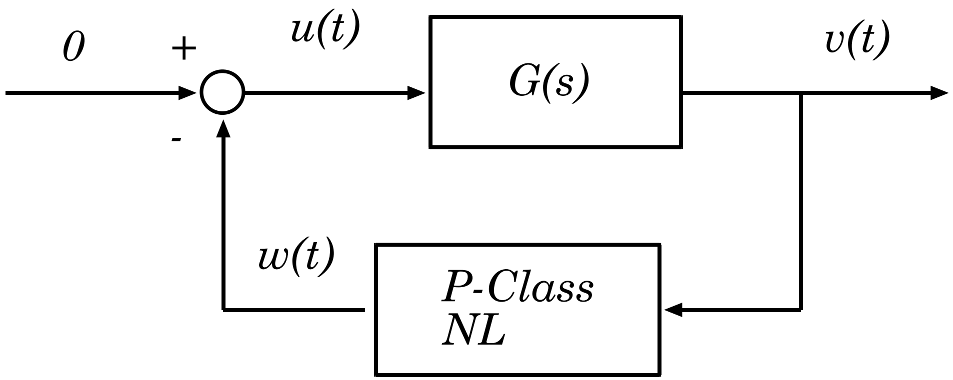
Asymptotic Hyperstability: The above feedback system is asymptotically hyperstable if

1. It is hyperstable

2. For all signals $|w(t)| < \infty$ (I.e. bounded output of any P-class nonlinearity), and any state space realization of *G(s)*,

$$\lim_{t \to \infty} x(t) = 0$$

CT Hyperstability Theorems



Hyperstability Theorem: The above feedback system is hyperstable **iff** the transfer function $G(s)$ of the LTI block is **Positive Real.**

Asymptotical Hyperstability Theorem: The above feedback system is asymptotically hyperstable **iff** the transfer function $G(s)$ of the LTI block is **Strictly Positive Real.**

CT Positive Real TF

$$G(s) = C(sI - A)^{-1}B + D$$

Is **Positive Real** iff:

1. $G(s)$ does not have any unstable poles (I.e. no Re{s} > 0).

2. Any pole of $G(s)$ that is in the imaginary axis <u>does not repeat</u> and its associated residue (I.e. the coefficient appearing in the partial fraction expansion) is non-negative.

3. $$2\,\mathrm{Re}\{G(j\omega)\} = G(j\omega) + G^T(-j\omega) \geq 0$$

for all real ω's for which $s = j\,\omega$ is not a pole of $G(s)$

Strictly Positive Real (SPR) TF

$$G(s) = C(sI - A)^{-1}B + D$$

Is **Strictly Positive Real (SPR)** iff:

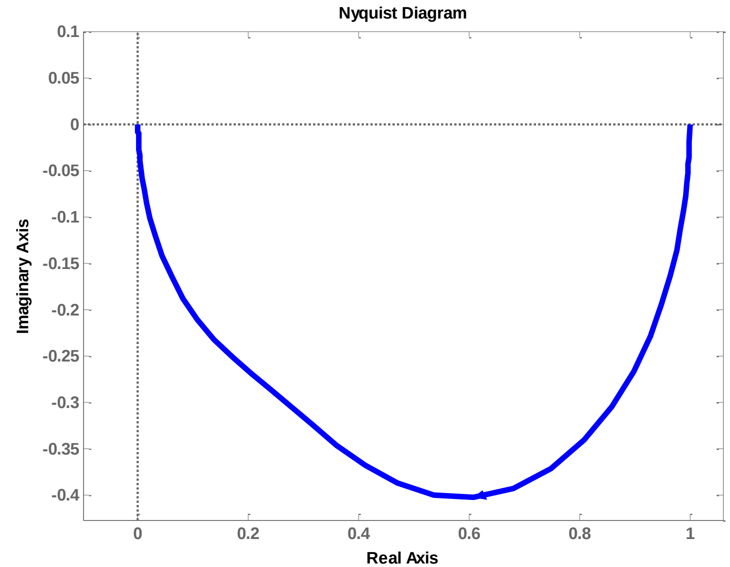
1. All poles of $G(s)$ are asymptotically stable.

2. $2\,\mathrm{Re}\{G(j\omega)\} = G(j\omega) + G^T(-j\omega) > 0$

   for all ω, $0 \le \omega < \infty$

Example:

$$G(s) = \frac{s+1}{s^2 + 3s + 1}$$



Strictly Positive Real (SPR) TF

For scalar rational transfer functions

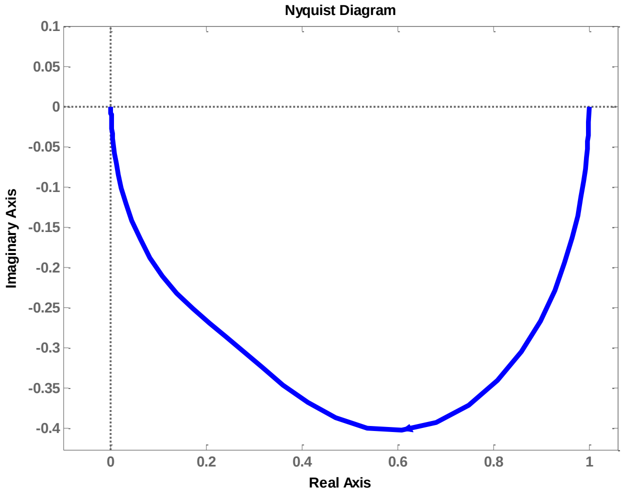
$$G(s) = \frac{B(s)}{A(s)}$$

1. All poles of $G(s)$ are asymptotically stable.

2. $\text{Re}\{G(j\omega)\} > 0$ for all ω, $0 \leq \omega < \infty$

Note:

A necessary (but not sufficient) condition for $G(s)$ to be SPR is that its relative degree must be less than or equal to 1.



Kalman Yakubovich Popov Lemma

$$G(s) = C(sI - A)^{-1}B + D$$

Is **Strictly Positive Real (SPR)** if and only if

- there exist a symmetric and positive definite matrix P,
- matrices L and K,
- and a constant ε > 0 such that

$$\begin{aligned} A^T P + PA &= -L^T L - \epsilon P \\ B^T P - C &= -K^T L \\ D + D^T &= K^T K \end{aligned}$$

Kalman Yakubovich Popov Lemma

$$G(s) = C(sI - A)^{-1}B$$

Is **Strictly Positive Real (SPR) iff** there exist symmetric and positive definite matrices P and Q, such that:

$$\begin{aligned} A^T P + PA &= -Q \\ B^T P &= C \end{aligned}$$

SPR TF implies Possitivity

Let $G(s) = C(sI - A)^{-1}B + D$ be SPR

Then there exist positive definite functions

$V(x) \succ 0 \qquad \lambda_1(x) \succ 0$

and a positive semi-definite function $\lambda_2(x,u) \succeq 0$

Such that the input *u(t)* output *y(t)* pair satisfies

$$\boxed{\begin{aligned} \int_0^t y^T u \, d\tau &= V(x(t)) - V(x(0)) + \int_0^t (\lambda_1(x) + \lambda_2(x,u))\, d\tau \\ &\geq -\gamma_o^2 \qquad\qquad \gamma_o^2 = V(x(0)) \end{aligned}}$$

SPR TF implies Passivity

Proof: We consider a strictly causal transfer function

$$G(s) = C(sI - A)^{-1}B$$

which is SPR, with state space realization

$$\begin{aligned} \frac{d}{dt}x &= A\,x + B\,u \\ v &= C\,x \end{aligned}$$

By the Kalman Yakubovich, Popov lemma, there exist symmetric and positive definite matrices P and Q, such that

$$\begin{aligned} A^T P + PA &= -Q \\ B^T P &= C \end{aligned}$$

SPR TF implies Passivity

Proof: Define the PD function $V(x) = \frac{1}{2} x^T P x$

and compute:

$$
\begin{aligned}
2\,\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\
&= (Ax + Bu)^T P x + x^T P (Ax + Bu) \\
&= x^T \left[\underbrace{A^T P + P A}_{-Q} \right] x + 2 u^T \underbrace{\underbrace{B^T P}_{C} x}_{v}
\end{aligned}
$$

by the Kalman Yakubovich, Popov lemma.

$$
\begin{aligned}
A^T P + P A &= -Q \\
B^T P &= C
\end{aligned}
$$

SPR TF implies Passivity

Proof: Thus, since $v = C\,x$

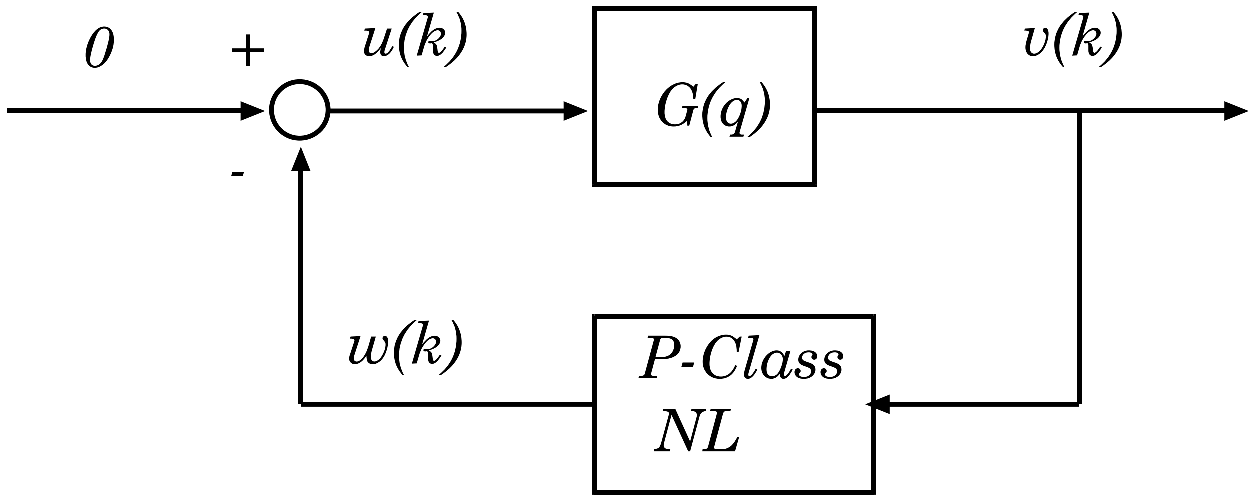
$$u^T v = \dot{V} + \frac{1}{2} x^T Q x$$

Define the PD function $\lambda_1(x) = \frac{1}{2} x^T Q x$ and integrate

$$\begin{aligned} \int_0^t u^T v \, d\tau &= \int_0^t \dot{V} \, d\tau + \int_0^t \lambda_1(x) \, d\tau \\ &= V(x(t)) - V(x(0)) + \int_0^t \lambda_1(x) \, d\tau \end{aligned}$$

DT Hyperstability Theory

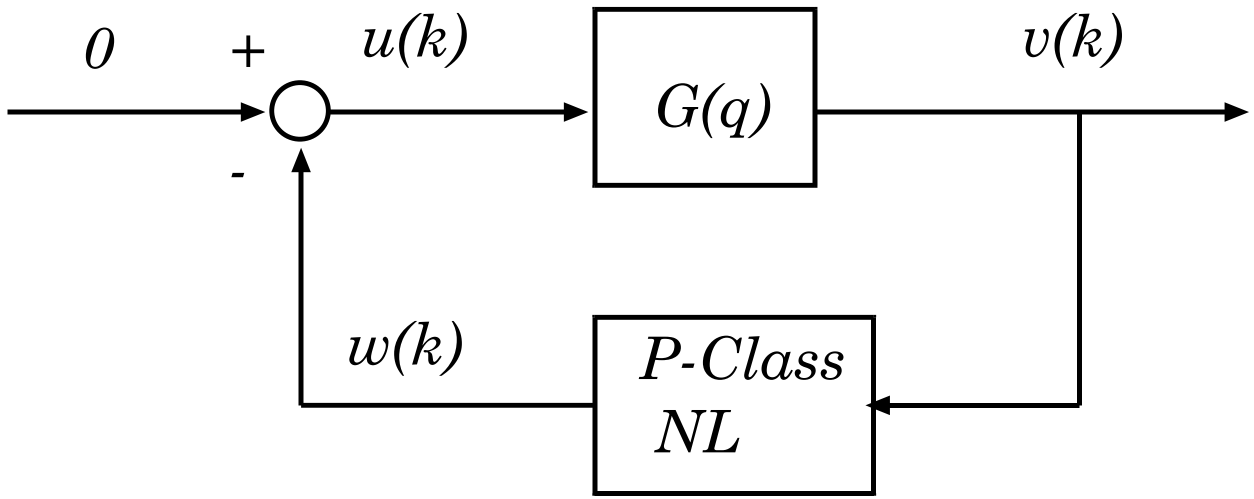
$$G(z) = C(zI - A)^{-1}B + D$$



- State space description of the LTI Block:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ v(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

DT Hyperstability Theory

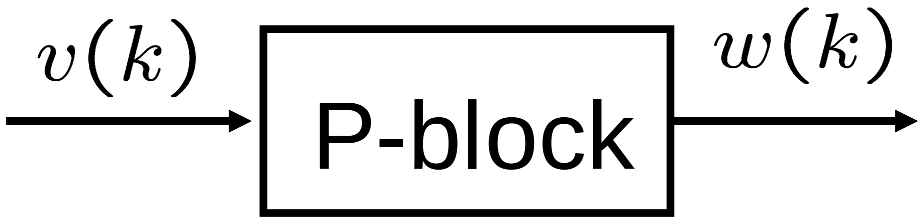


- P-class nonlinearity: (passive nonlinearities)

$$\boxed{\sum_{j=0}^{k} w^T(j)v(j) \geq -\gamma_o^2 \qquad \forall\, k \geq 0}$$

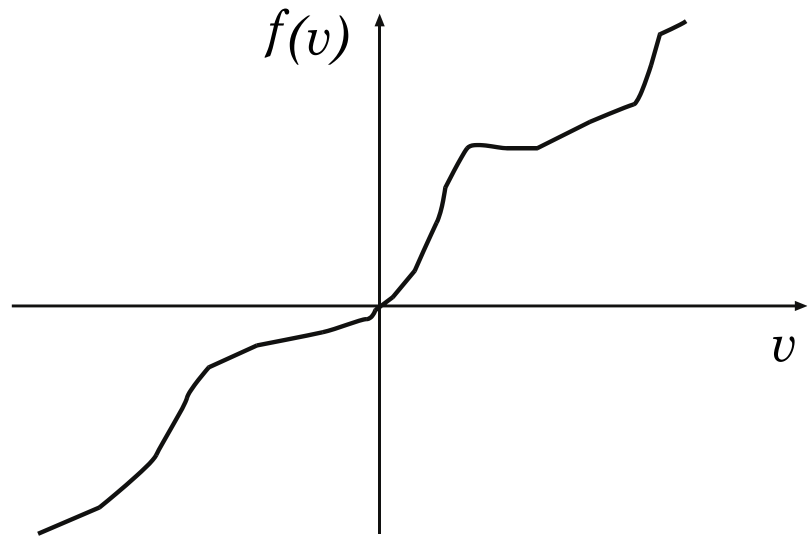
Where γ_o is a bounded constant.

Example: Static nonlinearity:



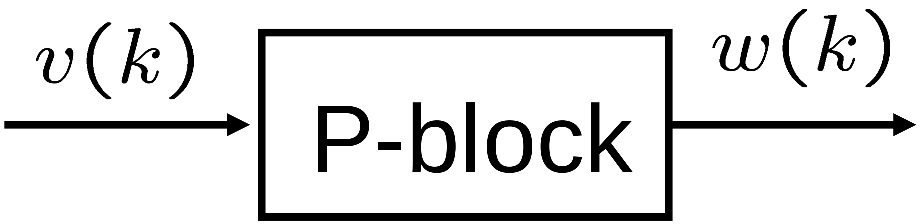
$$w = f(v)$$

$$v\, f(v) \geq 0$$



$$\sum_{j=0}^{k} w^T(j)v(j) = \sum_{j=0}^{k} \underbrace{f(v(j))v(j)}_{\geq 0} \geq 0 > -\gamma_o^2$$

Example: Dynamic P-class block



$$\begin{cases} \tilde{\theta}(k) = \tilde{\theta}(k-1) + F\,\phi(k)v(k) \\ w(k) = \phi^T(k)\tilde{\theta}(k) \end{cases}$$

$$F = F^T \succ 0$$

$$\phi(k) \in \mathcal{R}^n$$

$$\tilde{\theta}(-1) \in \mathcal{R}^n$$

$$\|\tilde{\theta}(-1)\| < \infty$$

$$\|\phi(k)\| < \infty$$

Example: Dynamic P-class block

$$w(k) = \phi^T(k)\tilde{\theta}(k) \qquad \tilde{\theta}(k) = \tilde{\theta}(k-1) + F\,\phi(k)v(k)$$

$$\sum_{j=0}^{k} w(j)v(j) = \sum_{j=0}^{k} \phi^T(j)\tilde{\theta}(j)v(j)$$

$$= \sum_{j=0}^{k} \tilde{\theta}^T(j)\,\underbrace{[\phi(j)v(j)]}_{F^{-1}[\tilde{\theta}(j)-\tilde{\theta}(j-1)]}$$

$$= \sum_{j=0}^{k} \tilde{\theta}^T(j)F^{-1}\left[\tilde{\theta}(j) - \tilde{\theta}(j-1)\right]$$

$$= \sum_{j=0}^{k} \left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j-1)\right\}$$

Example: Dynamic P-class block

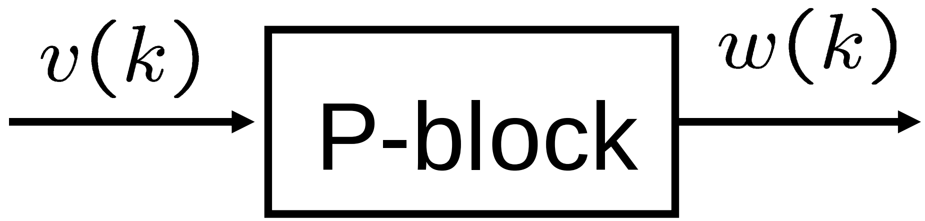
$$\sum_{j=0}^{k} w(j)v(j) = \sum_{j=0}^{k} \left\{ \tilde{\theta}^T(j) F^{-1} \tilde{\theta}(j) - \tilde{\theta}^T(j) F^{-1} \tilde{\theta}(j-1) \right\}$$

$$+ \frac{1}{2} \sum_{j=0}^{k} \tilde{\theta}^T(j-1) F^{-1} \tilde{\theta}(j-1) - \frac{1}{2} \sum_{j=0}^{k} \tilde{\theta}^T(j-1) F^{-1} \tilde{\theta}(j-1)$$

$$\sum_{j=0}^{k} w(j)v(j) = \frac{1}{2} \sum_{j=0}^{k} \left\{ \tilde{\theta}^T(j) F^{-1} \tilde{\theta}(j) - \tilde{\theta}^T(j-1) F^{-1} \tilde{\theta}(j-1) \right\}$$

$$+ \underbrace{\frac{1}{2} \sum_{j=0}^{k} \left[\tilde{\theta}(j) - \tilde{\theta}(j-1) \right]^T F^{-1} \left[\tilde{\theta}(j) - \tilde{\theta}(j-1) \right]}_{\geq 0}$$

Example: Dynamic P-class block

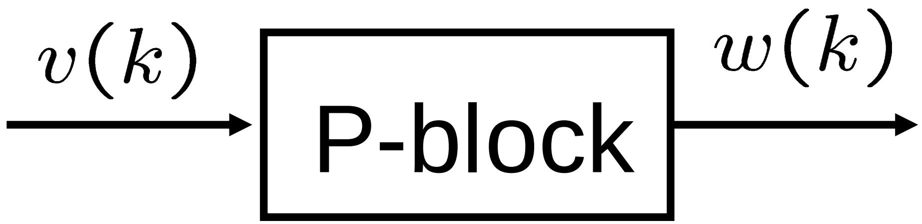


$$\sum_{j=0}^{k} w(j)v(j) \geq \quad \frac{1}{2}\sum_{j=0}^{k}\left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1)\right\}$$

$$= \frac{1}{2}\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k) - \underbrace{\frac{1}{2}\tilde{\theta}^T(-1)F^{-1}\tilde{\theta}(-1)}_{\gamma_o^2}$$

$$\geq -\gamma_o^2$$

Example: Dynamic P-class block



$$\begin{cases} \tilde{\theta}(k) = \tilde{\theta}(k-1) + F\phi(k-1)v(k) \\ w(k) = \phi^T(k-1)\tilde{\theta}(k) \end{cases}$$

$$\phi(k) \in \mathcal{R}^n$$

$$\tilde{\theta}(-1) \in \mathcal{R}^n$$

$$F = F^T \succ 0$$

$$\|\tilde{\theta}(-1)\| < \infty$$

$$\|\phi(k)\| < \infty$$

Example: Dynamic P-class block

$$w(k) = \phi^T(k-1)\tilde{\theta}(k) \qquad \tilde{\theta}(k) = \tilde{\theta}(k-1) + F\phi(k-1)v(k)$$

$$\sum_{j=0}^{k} w(j)v(j) = \sum_{j=0}^{k} \phi^T(j-1)\tilde{\theta}(j)v(j)$$

$$= \sum_{j=0}^{k} \tilde{\theta}^T(j)\underbrace{[\phi(j-1)v(j)]}_{F^{-1}\left[\tilde{\theta}(j)-\tilde{\theta}(j-1)\right]}$$

$$= \sum_{j=0}^{k} \tilde{\theta}^T(j)F^{-1}\left[\tilde{\theta}(j)-\tilde{\theta}(j-1)\right]$$

$$= \sum_{j=0}^{k} \left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j)-\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j-1)\right\}$$

Example: Dynamic P-class block

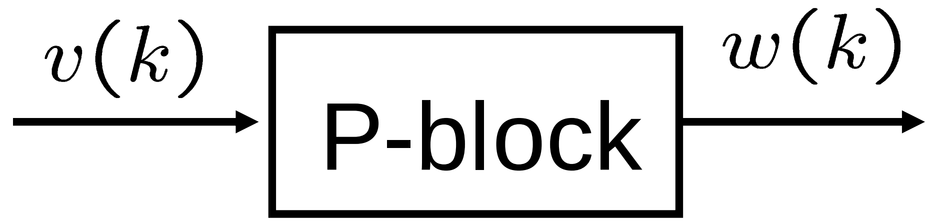
$$\sum_{j=0}^{k} w(j)v(j) = \sum_{j=0}^{k} \left\{ \tilde{\theta}^T(j) F^{-1} \tilde{\theta}(j) - \tilde{\theta}^T(j) F^{-1} \tilde{\theta}(j-1) \right\}$$

$$+ \frac{1}{2} \sum_{j=0}^{k} \tilde{\theta}^T(j-1) F^{-1} \tilde{\theta}(j-1) - \frac{1}{2} \sum_{j=0}^{k} \tilde{\theta}^T(j-1) F^{-1} \tilde{\theta}(j-1)$$

$$\sum_{j=0}^{k} w(j)v(j) = \frac{1}{2} \sum_{j=0}^{k} \left\{ \tilde{\theta}^T(j) F^{-1} \tilde{\theta}(j) - \tilde{\theta}^T(j-1) F^{-1} \tilde{\theta}(j-1) \right\}$$

$$+ \underbrace{\frac{1}{2} \sum_{j=0}^{k} \left[\tilde{\theta}(j) - \tilde{\theta}(j-1) \right]^T F^{-1} \left[\tilde{\theta}(j) - \tilde{\theta}(j-1) \right]}_{\geq 0}$$

Example: Dynamic P-class block



$$\sum_{j=0}^{k} w(j)v(j) \geq \frac{1}{2}\sum_{j=0}^{k}\left\{\tilde{\theta}^T(j)F^{-1}\tilde{\theta}(j) - \tilde{\theta}^T(j-1)F^{-1}\tilde{\theta}(j-1)\right\}$$

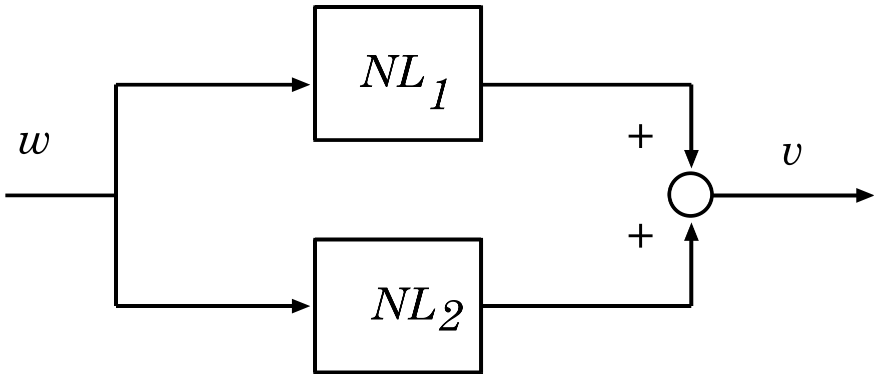
$$= \frac{1}{2}\tilde{\theta}^T(k)F^{-1}\tilde{\theta}(k) - \underbrace{\frac{1}{2}\tilde{\theta}^T(-1)F^{-1}\tilde{\theta}(-1)}_{\gamma_o^2}$$

$$\geq -\gamma_o^2$$

Examples of P-class NL

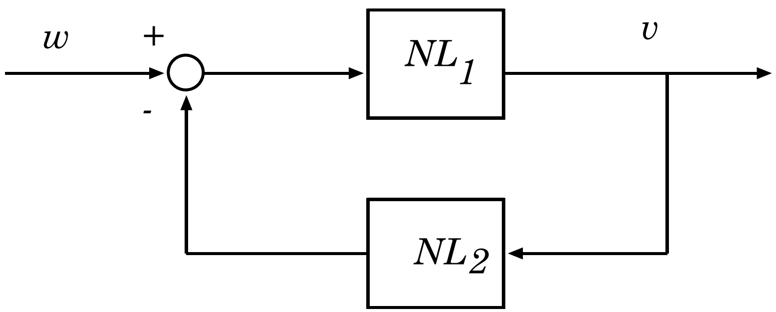
Lemma:

- The parallel combination of two P-class nonlinearities is also a P-class nonlinearity.



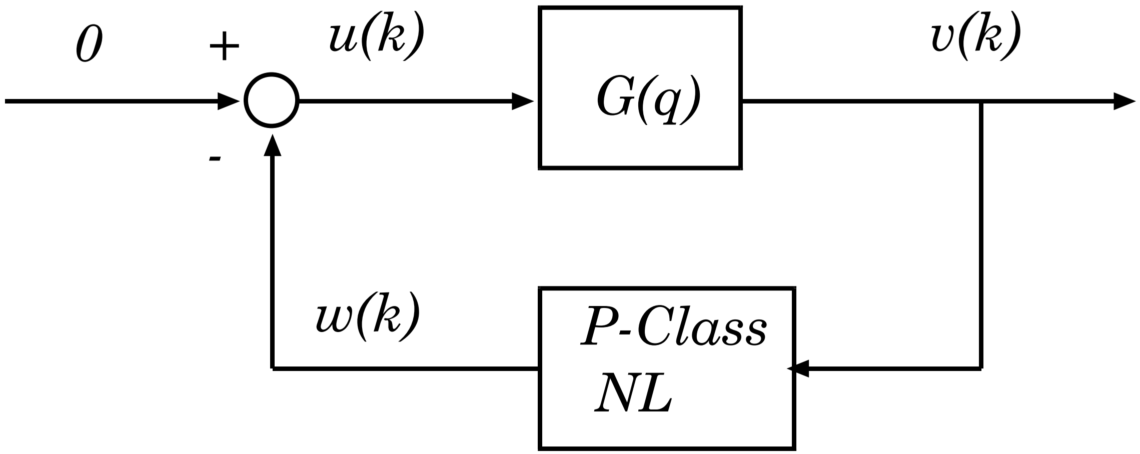
$$\sum_{j=0}^{k} w^T(j)v(j) \geq -\gamma_o^2$$

- The feedback combination of two P-class nonlinearities is also a P-class nonlinearity.



$$\sum_{j=0}^{k} w^T(j)v(j) \geq -\gamma_o^2$$

DT Hyperstability

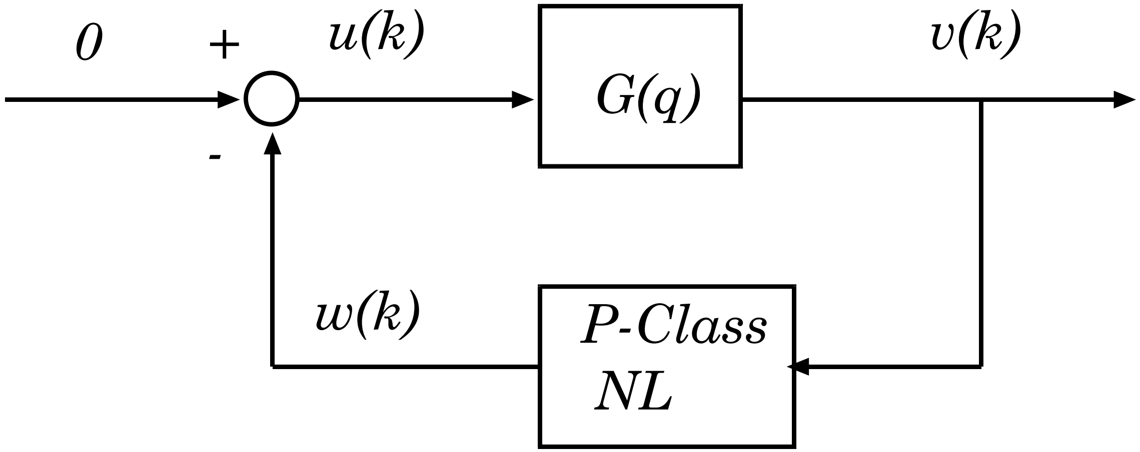


Hyperstability: The above feedback system is hyperstable if there exist positive bounded constants δ_1, δ_2 such that, for any state space realization of *G(q)*,

$$\|x(k)\| < \delta_1 \left[\|x(0)\| + \delta_2\right] \qquad \forall\, k \geq 0$$

<u>FOR ALL</u> P-class nonlinearities

DT Asymptotic Hyperstability



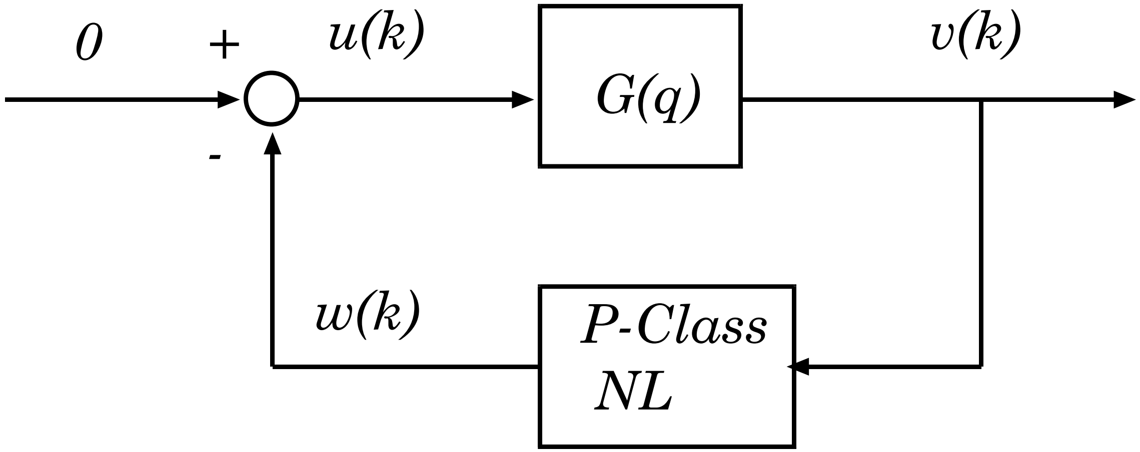
Asymptotic Hyperstability: The above feedback system is asymptotically hyperstable if

1. It is hyperstable

2. for any state space realization of $G(z)$,

$$\lim_{k\to\infty} x(k) = 0$$

DT Hyperstability Theorems



Hyperstability Theorem: The above feedback system is hyperstable **iff** the transfer function $G(z)$ of the LTI block is **Positive Real.**

Asymptotical Hyperstability Theorem: The above feedback system is asymptotically hyperstable **iff** the transfer function $G(z)$ of the LTI block is **Strictly Positive Real.**

Positive Real TF

$$G(z) = C(zI - A)^{-1}B + D$$

Is **Positive Real** iff:

1. $G(z)$ does not have any unstable poles (I.e. no $|z| > 1$).

2. Any pole of $G(z)$ that is in the unit circle does not repeat and its associated residue (i.e. the coefficient appearing in the partial fraction expansion) is non-negative.

3. $G(e^{j\omega}) + G^T(e^{-j\omega}) \succeq 0$

   for all $\omega \in [0, \pi]$ for which $z = e^{j\omega}$ is not a pole of $G(z)$

Strictly Positive Real (SPR) TF

$$G(z) = C(zI - A)^{-1}B + D$$

Is **Strictly Positive Real (SPR)** iff:

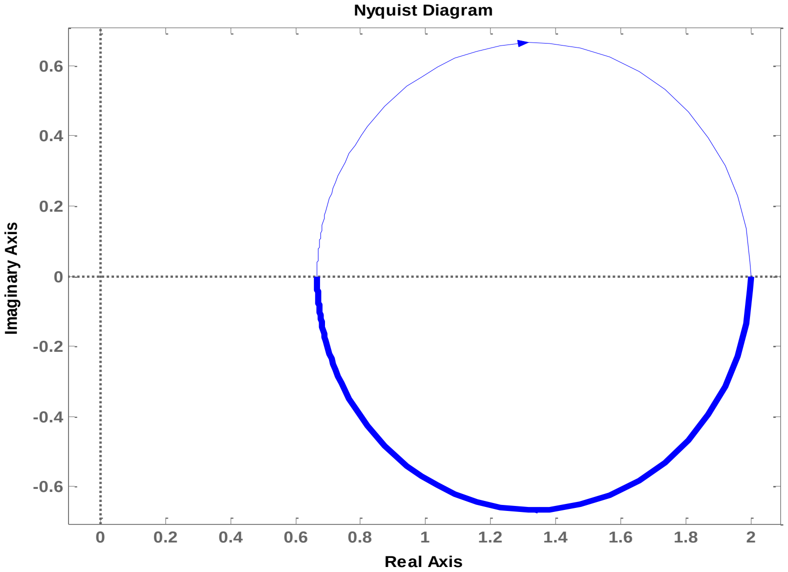
1. All poles of $G(z)$ are asymptotically stable.

2. $G(e^{j\omega}) + G^T(e^{-j\omega}) \succ 0$

   for all $0 \leq \omega \leq \pi$

Example:

$$G(z) = \frac{z}{z + 0.5}$$



Strictly Positive Real (SPR) TF

For scalar rational transfer functions

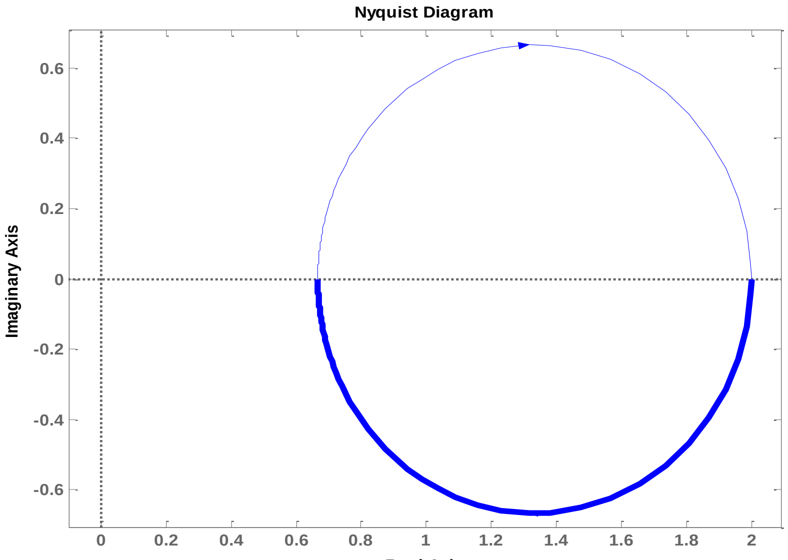
$$G(z) = \frac{B(z)}{A(z)}$$

1. All poles of $G(z)$ are asymptotically stable.

2. $\mathrm{Re}\{G(e^{j\omega})\} > 0$ for all ω, $0 \leq \omega \leq \pi$

Note:

A necessary (but not sufficient) condition for $G(z)$ to be SPR is that its relative degree must be 0.



Matrix Inequality Interpretation of SPR

The transfer function

$$G(z) = C(zI - A)^{-1}B + D$$

is **Strictly Positive Real (SPR)** if and only if

there exists $P \succ 0$ such that

$$\begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - D - D^T \end{bmatrix} \prec 0$$

SPR state-space realization fact

Theorem: If $G(z) = C(zI-A)^{-1}B + D$ is SPR, then

$$D + D^T \succ 0$$

Proof: Choose $P \succ 0$ such that

$$\begin{bmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B - D - D^T \end{bmatrix} \prec 0$$

Note that

$$B^T P B - D - D^T \prec 0$$

$$\Rightarrow D + D^T \succ B^T P B \succeq 0$$

■

SPR TF is P-class

Let $G(z) = C(zI - A)^{-1}B + D$ be SPR

Then there exist positive definite functions

$V(x) \succ 0 \qquad \lambda_1(x,u) \succ 0$

Such that any input *u(k)* output *y(k)* pair satisfies

$$\boxed{\begin{aligned}\sum_{j=0}^{k} y^T(j)u(j) &= V(x(k+1)) - V(x(0)) + \sum_{j=0}^{k} \lambda_1(x(j),u(j)) \\ &\geq -\gamma_0^2 \qquad\qquad \gamma_o^2 = V(x(0))\end{aligned}}$$

Shorthand notation

$$x(k) \to x_k$$

$$u(k) \to u_k$$

$$y(k) \to y_k$$

$$v(k) \to v_k$$

$$w(k) \to w_k$$

Proof

Let $G(z) = C(zI - A)^{-1}B + D$ be SPR

Choose $P \succ 0$ such that

$$\begin{bmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & B^T PB - D - D^T \end{bmatrix} \prec 0$$

Define the Lyapunov function

$$V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$$

and the function

$$\lambda_1(x,u) = -\frac{1}{2} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T PA - P & A^T PB - C^T \\ B^T PA - C & B^T PB - D - D^T \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \succ 0$$

Proof

$$V(x_{k+1}) - V(x_k) = \frac{1}{2}(Ax_k + Bu_k)^T P(Ax_k + Bu_k) - \frac{1}{2}x_k^T P x_k$$

$$= \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - D - D^T \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$+ \frac{1}{2}\left[u_k^T C x_k + u_k^T D u_k + u_k^T D^T u_k + x_k^T C^T u_k\right]$$

$$= -\lambda_1(x_k, u_k) + (Cx_k + Du_k)^T u_k$$

$$= -\lambda_1(x_k, u_k) + y_k^T u_k$$

Proof

From the previous slide

$$V(x_{k+1}) - V(x_k) = -\lambda_1(x_k, u_k) + y_k^T u_k$$

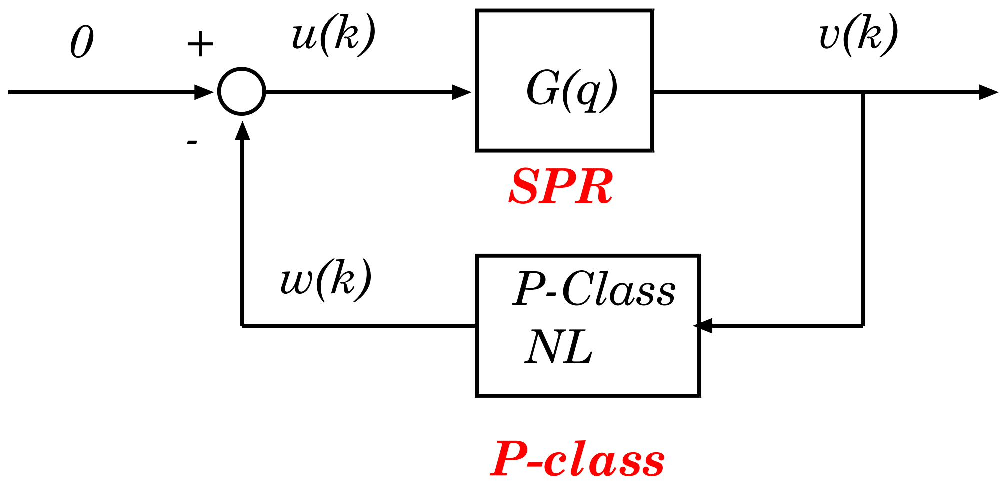
$$\Rightarrow \quad y_k^T u_k = V(x_{k+1}) - V(x_k) + \lambda_1(x_k, u_k)$$

Summing both sides of the equation yields

$$\sum_{j=0}^{k} y_j^T u_j = V(x_{k+1}) - V(x_0) + \sum_{j=0}^{k} \lambda_1(x_j, u_j)$$

$\blacksquare$

Proof of the sufficiency part of the Asymptotic Hyperstability Theorem - Discrete Time

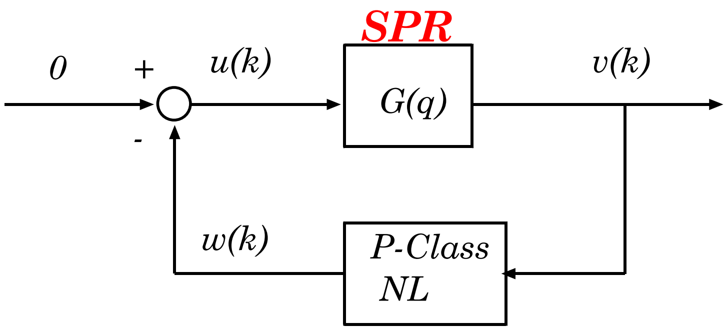


- Since the nonlinearity is P-class, $\sum_{j=0}^{k} w_j^T v_j \geq -\gamma_1^2$

- Since LTI block is SPR, we can use the choose $P \succ 0$ such that

$$\begin{bmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B - D - D^T \end{bmatrix} \prec 0$$

Hyperstability



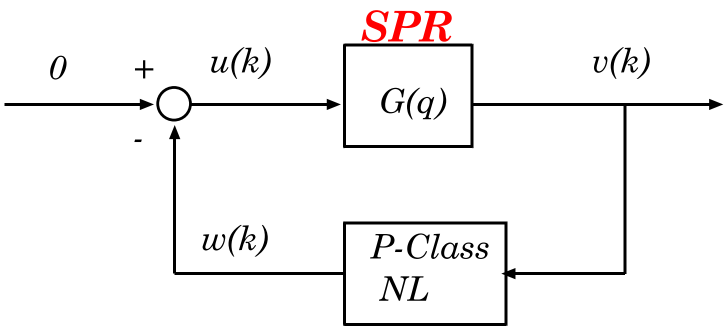
From the previous proof (SPR TF is P-class), we have

$$\sum_{j=0}^{k} v_j^T u_j = V(x_{k+1}) - V(x_0) + \sum_{j=0}^{k} \lambda_1(x_j, u_j)$$

where

$$V(x_k) = \frac{1}{2} x_k^T P x_k \succ 0$$

$$\lambda_1(x,u) = -\frac{1}{2} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B - D - D^T \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \succ 0$$



Hyperstability

$$G(z) = C(zI - A)^{-1}B + D$$

Rearranging terms,

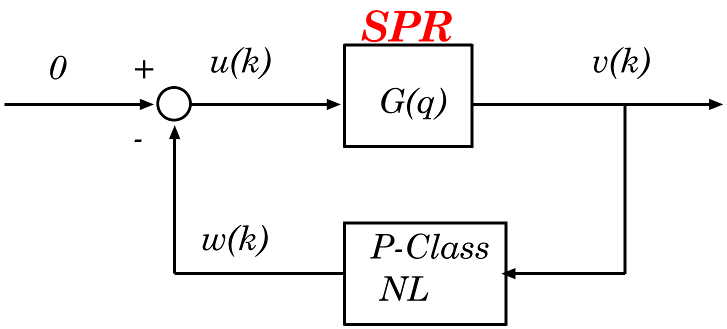
$$V(x_{k+1}) = V(x_0) + \sum_{j=0}^{k} v_j^T u_j - \sum_{j=0}^{k} \lambda_1(x_j, u_j)$$

From the P-class nonlinearity:

$$\sum_{j=0}^{k} w_j^T v_j \geq -\gamma_1^2 \quad \Rightarrow \quad \sum_{j=0}^{k} v_j^T u_j \leq \gamma_1^2$$

Therefore,

$$V(x_{k+1}) \leq V(x_0) + \gamma_1^2 - \underbrace{\sum_{j=0}^{k} \lambda_1(x_j, u_j)}_{\geq 0} \leq V(x_0) + \gamma_1^2$$



Hyperstability

$$G(z) = C(zI - A)^{-1}B + D$$

From the previous slide

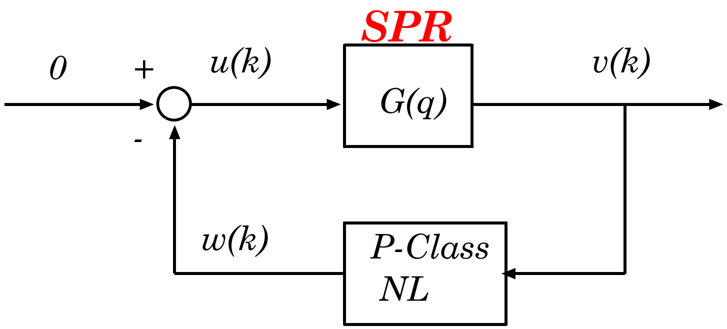
$$V(x_k) \leq V(x_0) + \gamma_1^2$$

$$\Rightarrow \quad \frac{1}{2}x_k^T P x_k \leq \frac{1}{2}x_0^T P x_0 + \gamma_1^2$$

$$\Rightarrow \quad \lambda_{min}(P)\|x_k\|^2 \leq \lambda_{max}(P)\|x_0\|^2 + 2\gamma_1^2$$

$$\Rightarrow \quad \|x_k\|^2 \leq \frac{\lambda_{max}(P)}{\lambda_{min}(P)}\left(\|x_0\|^2 + \frac{2}{\lambda_{max}(P)}\gamma_1^2\right)$$

Therefore, the feedback system is hyperstable



Asymptotic Hyperstability

$$G(z) = C(zI - A)^{-1}B + D$$

$$0 \leq \; V(x_{k+1}) \leq V(x_0) + \gamma_1^2 - \sum_{j=0}^{k} \lambda_1(x_j, u_j)$$

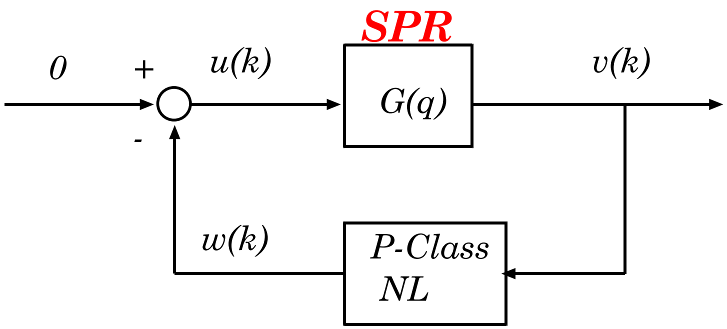
$$\Rightarrow \quad \underbrace{\sum_{j=0}^{k} \lambda_1(x_j, u_j)} \leq V(x_0) + \gamma_1^2$$

- monotonic nondecreasing sequence in k
- bounded above

$$\Rightarrow \quad \lim_{k\to\infty} \lambda_1(x_k, u_k) = 0 \quad \Rightarrow \quad \lim_{k\to\infty} x_k = 0, \; \lim_{k\to\infty} u_k = 0$$

Therefore, the feedback system is asymtotically hyperstable

■



Additional Result

$$G(z) = C(zI - A)^{-1}B + D$$

We have already shown that

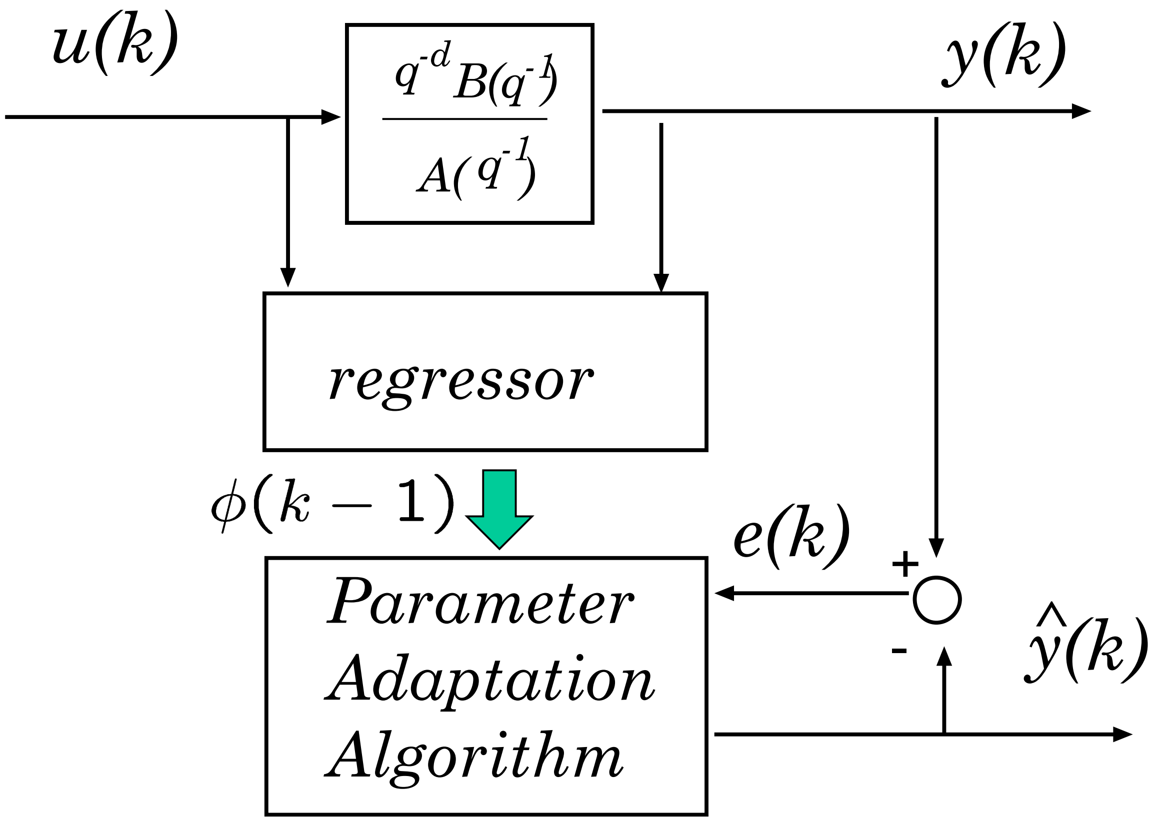
$$\lim_{k\to\infty} x_k = 0,$$
$$\lim_{k\to\infty} u_k = 0$$

From this we see that

$$\lim_{k\to\infty} v_k = \lim_{k\to\infty} (Cx_k + Du_k) = 0$$
$$\lim_{k\to\infty} w_k = \lim_{k\to\infty} (-u_k) = 0$$

$$\boxed{\text{Therefore, } x(k),\ u(k),\ v(k), \text{ and } w(k) \text{ converge to } 0}$$

Stability analysis of Series-parallel ID



Series-Parallel ID Dynamics (review)

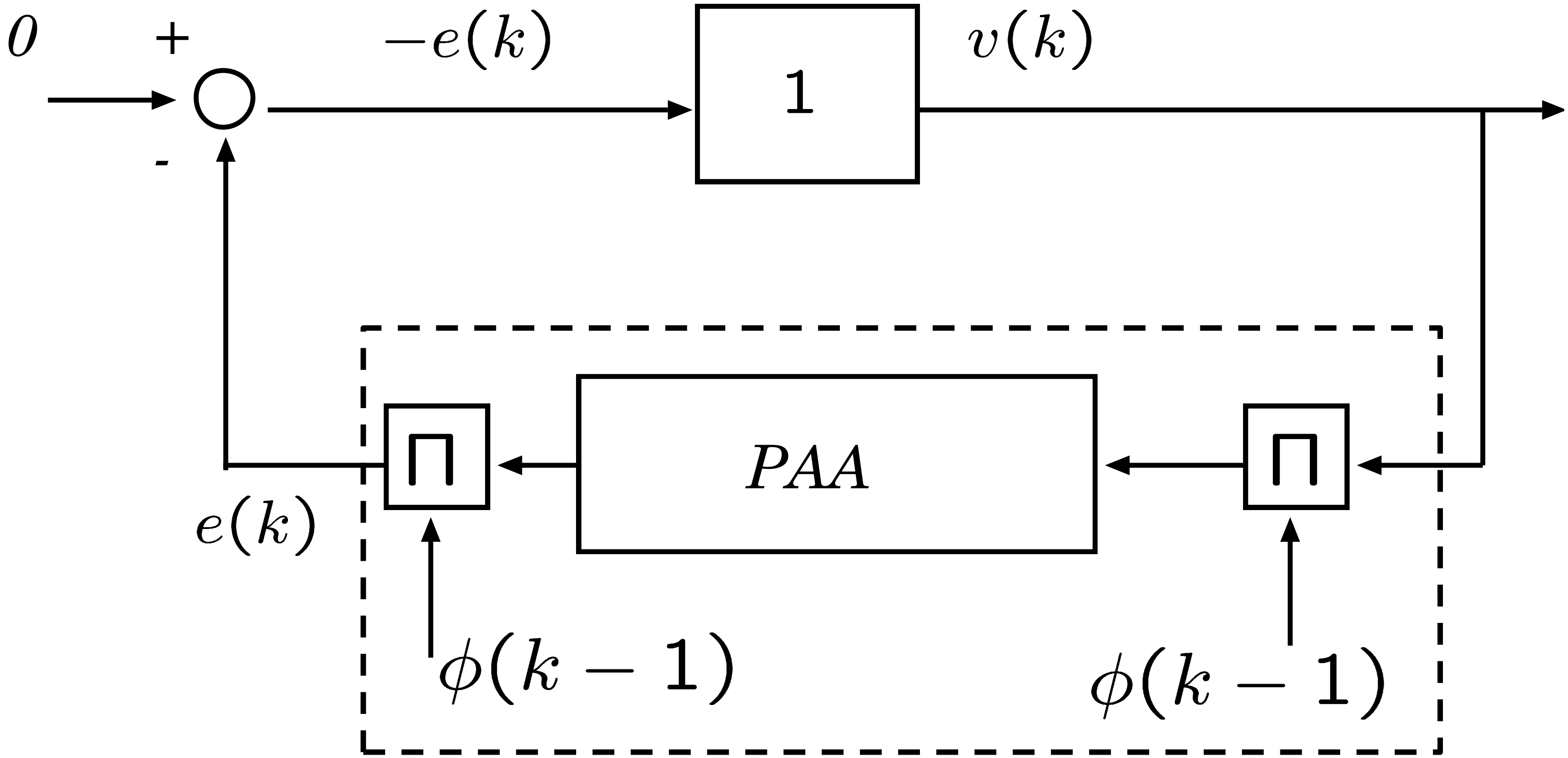
a-posteriori error: $e(k) = y(k) - \widehat{y}(k)$

$$e(k) = \phi^T(k-1)\tilde{\theta}(k)$$

Parameter error update law: $\tilde{\theta}(k) = \theta - \widehat{\theta}(k)$

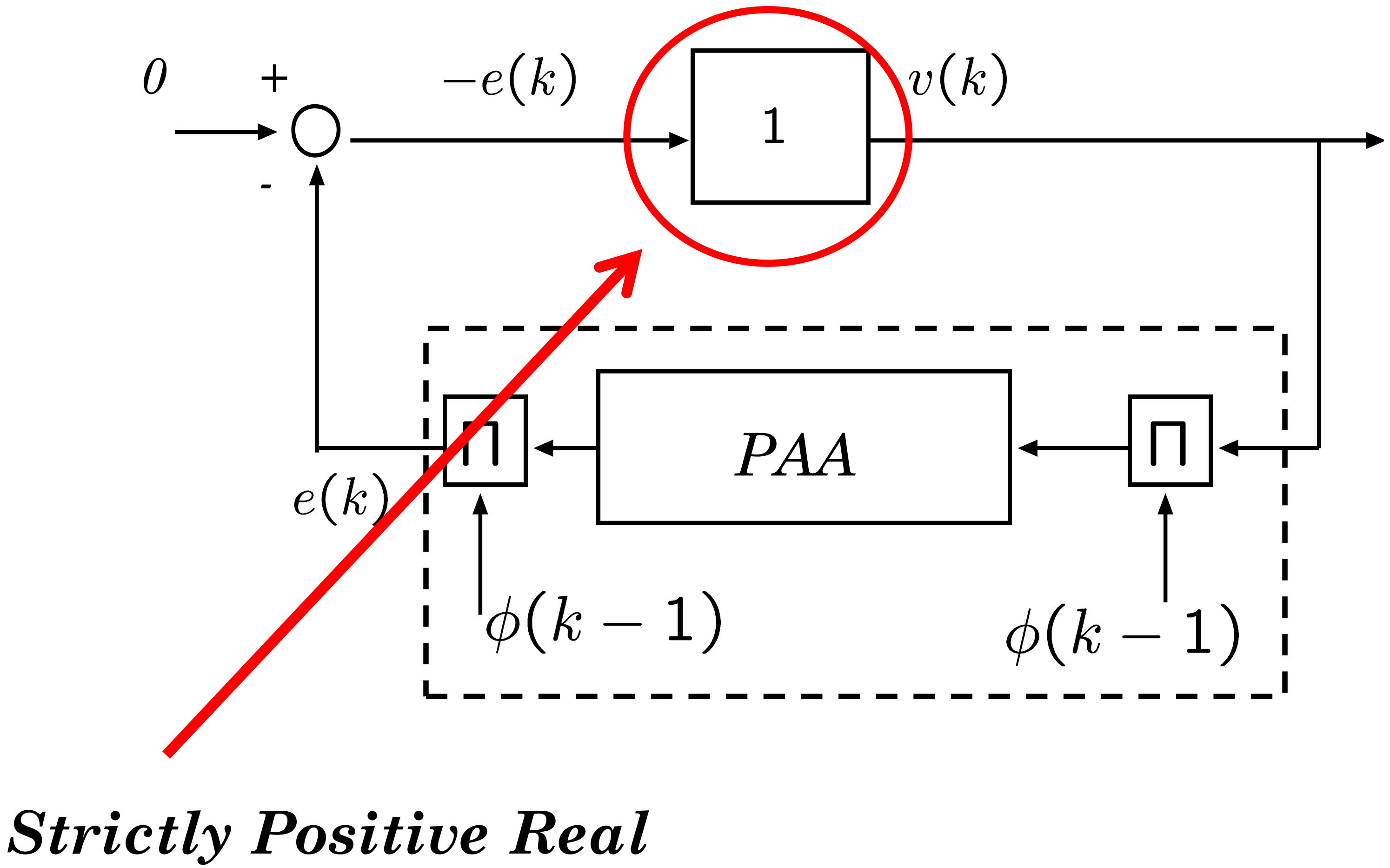
$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Series-Parallel ID Dynamics (review)

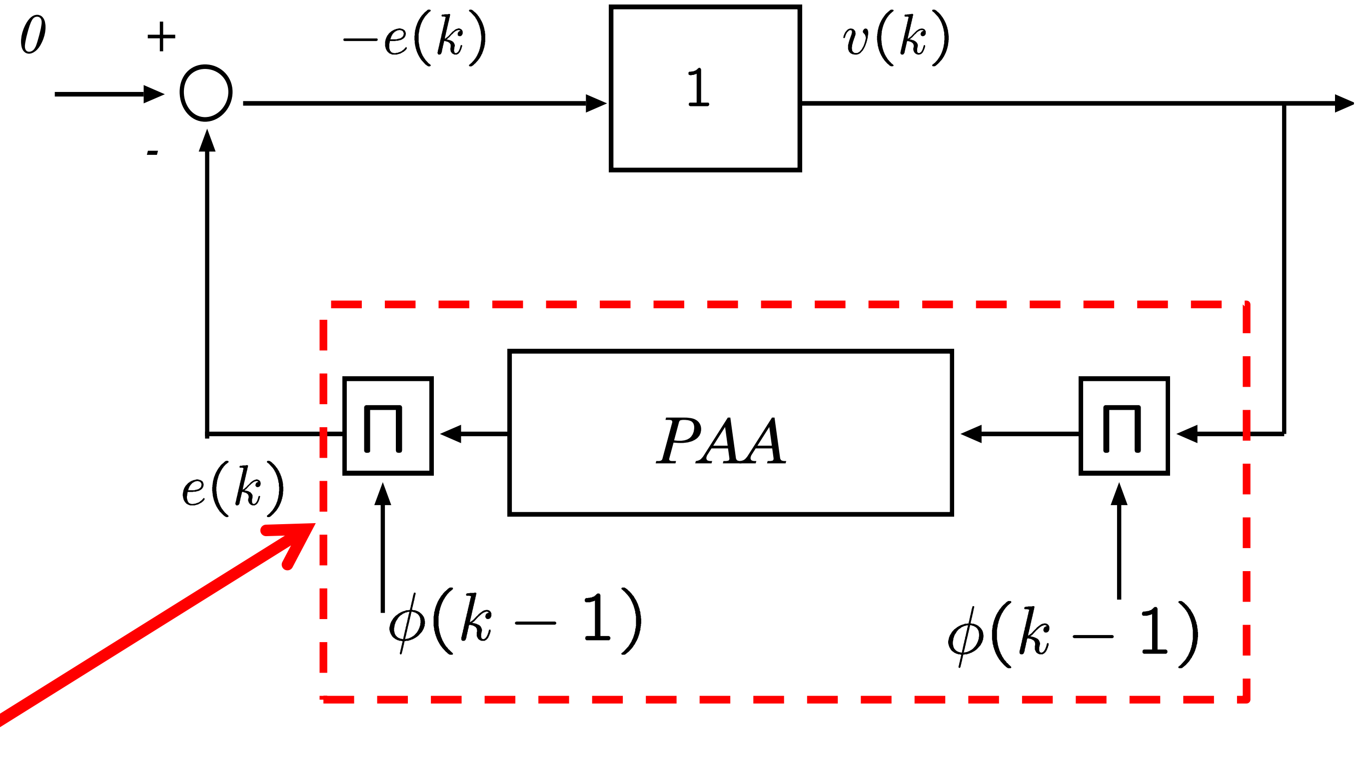


PAA: $$\tilde{\theta}(k) = \tilde{\theta}(k-1) + F\phi(k-1)v(k)$$

Stability analysis of Series-parallel ID

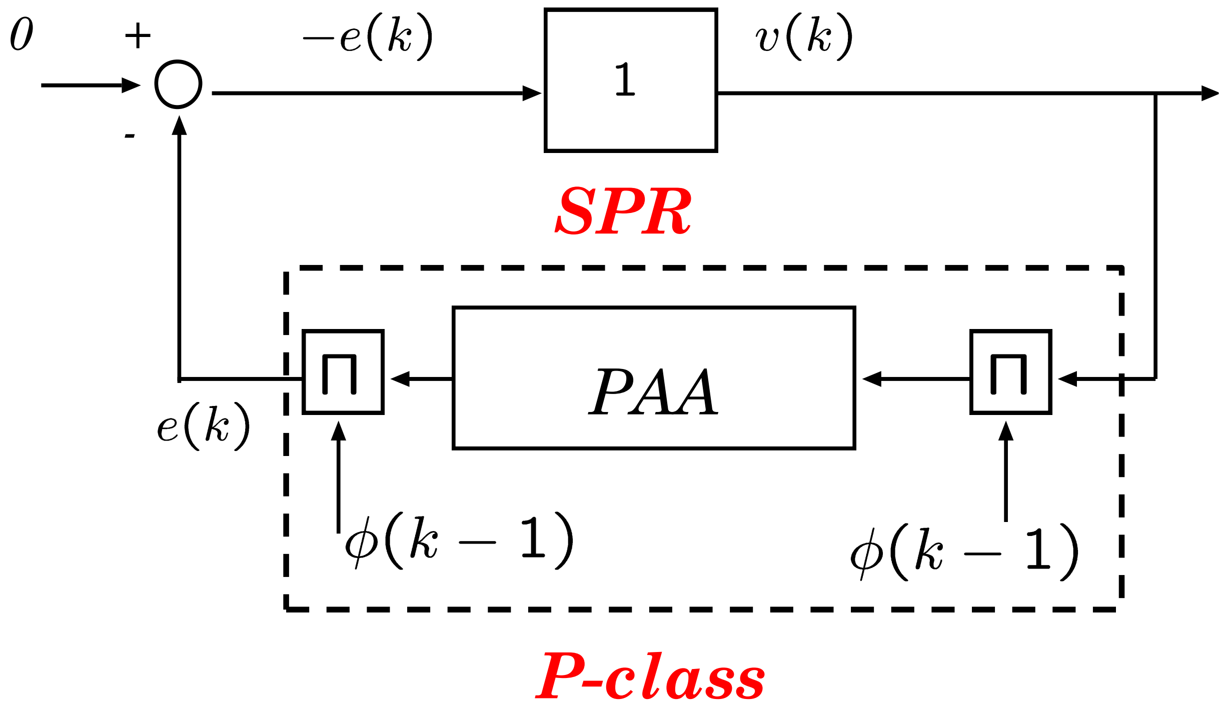


Stability analysis of Series-parallel ID



$$\left\{\begin{array}{l}\tilde{\theta}(k)=\tilde{\theta}(k-1)+F\,\phi(k-1)v(k)\\[1em] e(k)=\phi^{T}(k-1)\tilde{\theta}(k)\end{array}\right. \quad \Rightarrow \quad \sum_{j=0}^{k} e^{T}(j)v(j)\geq-\gamma_{o}^{2}$$

Stability analysis of Series-parallel ID

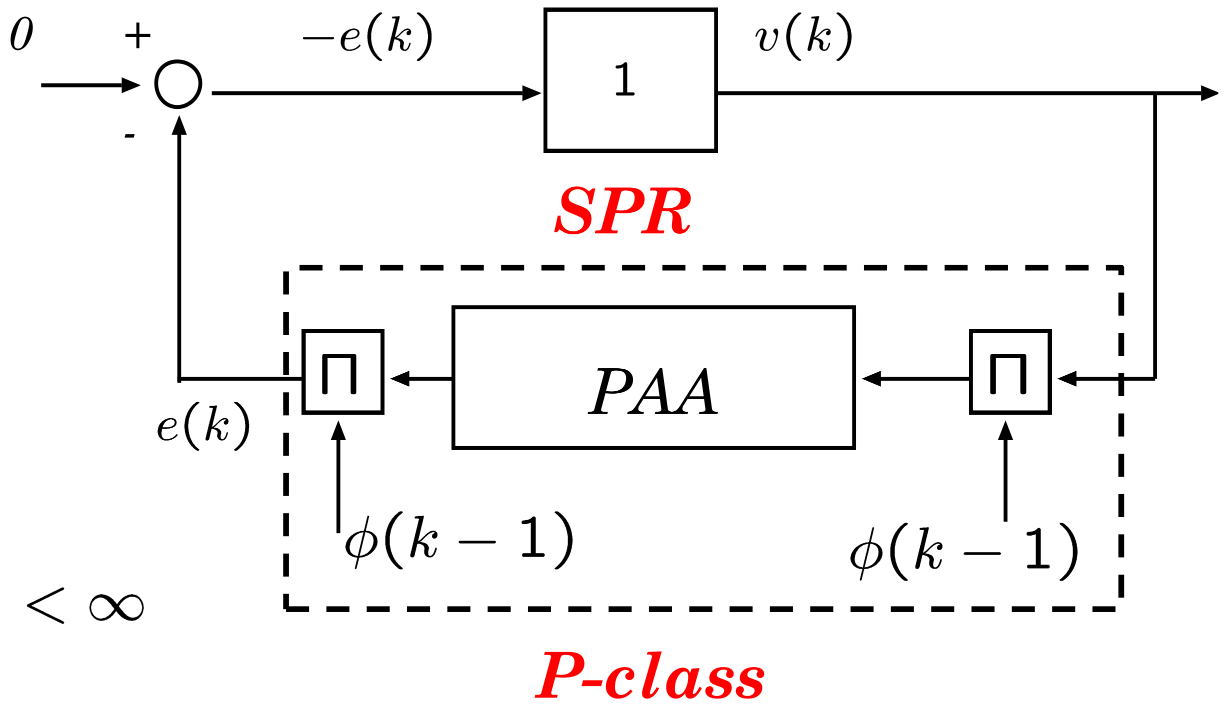


By the sufficiency portion of Hyperstability Theorem:

$$|v(k)| < \infty$$

$$|e(k)| < \infty$$

Stability analysis of Series-parallel ID



$$|e(k)| < \infty$$

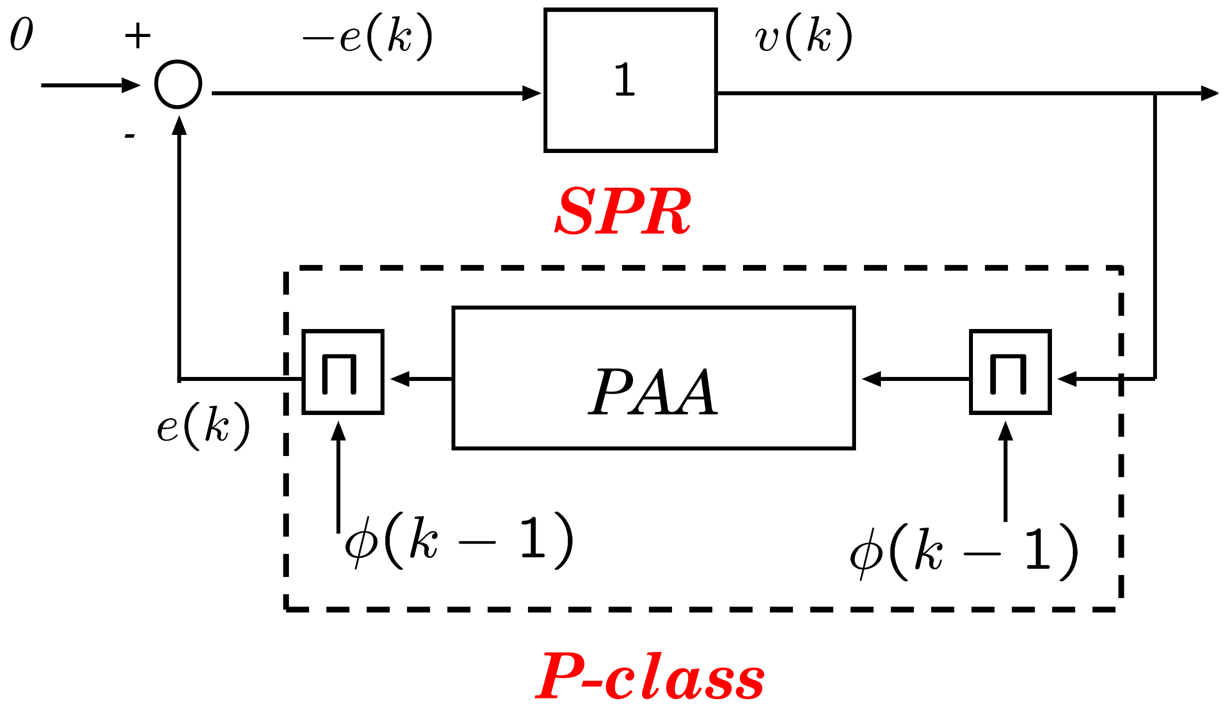
By the sufficiency portion of Asymptotic Hyperstability Theorem:

$$|v(k)| \to 0$$

$$|e(k)| \to 0$$

Q.E.D.

Stability analysis of Series-parallel ID



By the sufficiency portion of Asymptotic Hyperstability Theorem:

$$|v(k)| \longrightarrow 0$$

$$|e(k)| \longrightarrow 0$$

How to we implement the PAA?

a-posteriori error & PAA:

$$\left.\begin{aligned} e(k) &= \phi^T(k-1)\tilde{\theta}(k) \\ \tilde{\theta}(k) &= \tilde{\theta}(k-1) - F\,\phi(k-1)e(k) \end{aligned}\right\} \begin{array}{l} \mathit{Static} \\ \mathit{coupling} \end{array}$$

Solution: Use the a-priori error

$$e^o(k) = \phi^T(k-1)\tilde{\theta}(k-1)$$

How to we implement the PAA?

a-posteriori estimate & PAA:

$$\left.\begin{aligned} e(k) &= y(k) - \phi^T(k-1)\widehat{\theta}(k) \\ \widehat{\theta}(k) &= \widehat{\theta}(k-1) + F\phi(k-1)e(k) \end{aligned}\right\} \begin{aligned} &Static \\ &coupling \end{aligned}$$

Solution: Use the a-priori error

$$e^o(k) = y(k) - \phi^T(k-1)\widehat{\theta}(\underline{k-1})$$

$$= \phi^T(k-1)\widetilde{\theta}(\underline{k-1})$$

How to we implement the PAA?

$$\tilde{\theta}(k) = \tilde{\theta}(k-1) - F\,\phi(k-1)e(k)$$

Multiply by $\phi^T(k-1) = \phi^T_{k-1}$

$$\underbrace{\phi^T_{k-1}\tilde{\theta}(k)}_{e(k)} = \underbrace{\phi^T_{k-1}\tilde{\theta}(k-1)}_{e^o(k)} - \phi^T_{k-1}F\,\phi_{k-1}e(k)$$

$$e(k) = e^o(k) - \phi^T_{k-1}F\,\phi_{k-1}e(k)$$

Therefore,

$$e(k) = \frac{e^o(k)}{1 + \phi^T(k-1)F\,\phi(k-1)}$$

How we implement the PAA

1. $$e^o(k) = y(k) - \phi^T(k-1)\widehat{\theta}(k-1)$$

2. $$e(k) = \frac{e^o(k)}{1 + \phi^T(k-1)F\,\phi(k-1)}$$

3. $$\widehat{\theta}(k) = \widehat{\theta}(k-1) + F\phi(k-1)e(k)$$

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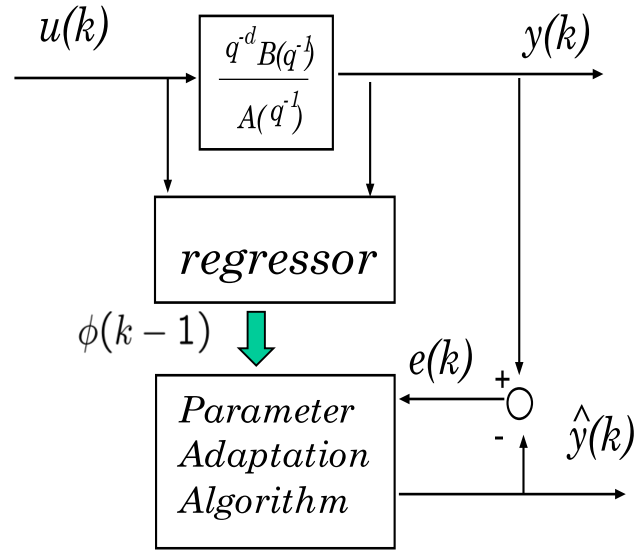

We have shown that

$$e(k) \to 0$$

Now we will show that

$$e^o(k) \to 0$$

Under the following assumptions:

$$|u(k)| < \infty \qquad A(q^{-1}) \text{ is anti-Schur}$$

Since $$y(k) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(k) \quad \Rightarrow \quad |y(k)| < \infty$$

Since $$\phi(k-1) = \begin{bmatrix} y(k-1) \\ \vdots \\ u(k-d) \\ \vdots \end{bmatrix} \quad \Rightarrow \quad \|\phi(k-1)\| < \infty$$

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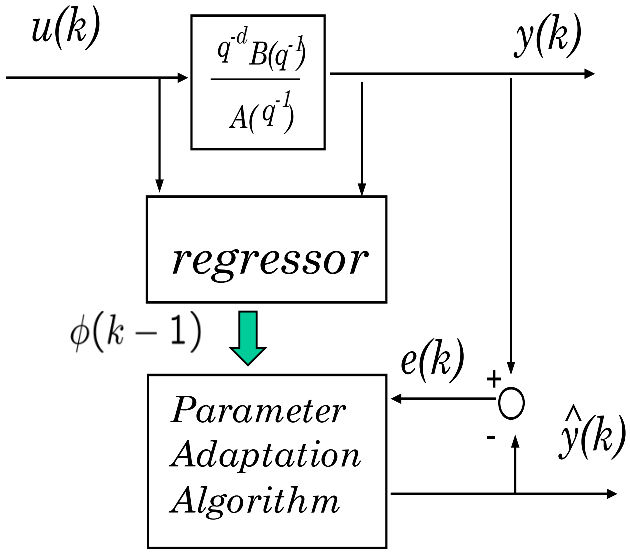

Thus, we know that

$$e(k) \to 0$$

$$\|\phi(k-1)\| < \infty$$

Remember that

$$e(k) = \frac{e^o(k)}{1 + \phi^T(k-1)F\,\phi(k-1)}$$

$$\Rightarrow \quad e^o(k) = \underbrace{e(k)}_{\to 0}\underbrace{\{1 + \phi^T(k-1)F\,\phi(k-1)\}}_{<\infty}$$

$$\Rightarrow \quad e^o(k) \to 0$$

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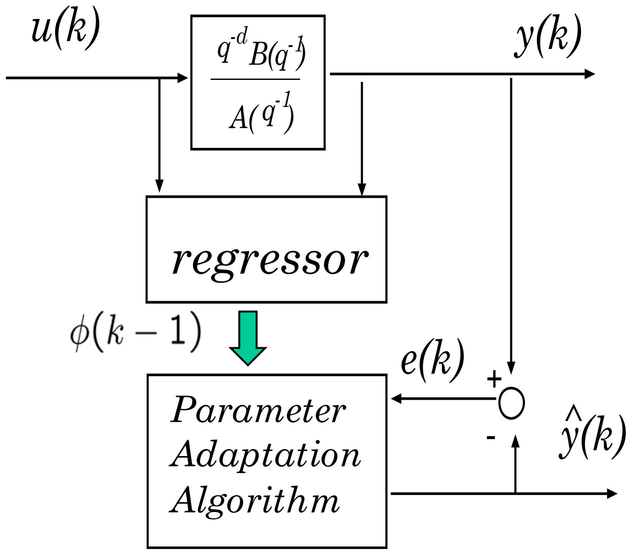

We have shown that

$e(k) \to 0 \quad e^o(k) \to 0$

$\|\phi(k-1)\| < \infty$

What about the parameter error $\tilde{\theta}(k)$?

since

$$\underbrace{e^o(k)}_{\to 0} = \phi^T(k-1)\tilde{\theta}(k-1) \quad \Rightarrow \quad |\phi^T(k)\tilde{\theta}(k)| \to 0$$

However, this **<u>does not imply</u>** that the parameter error goes to zero

We need to impose another condition on $u(k)$ to guarantee that the parameter error goes to zero. (**<u>persistence of excitation</u>**)