ME 233 – Advanced Control II

Lecture 17

Minimum Variance Regulator

Tony Kelman

UC Berkeley

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Outline

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Model Form

We consider a state space model of the form

$$x(k+1) = \hat{A}x(k) + \hat{B}u(k) + \hat{B}_w w(k)$$
$$y(k) = \hat{C}x(k) + v(k)$$

where

- $u(k)$ is the **scalar** control signal
- $y(k)$ is the **scalar** measurement signal
- $w(k)$ is the input noise
  (white, zero-mean, $E\{w(k)w^T(k)\} = W$)
- $v(k)$ is the measurement noise
  (white, zero-mean, $E\{v(k)v^T(k)\} = V$)
- $E\{w(k)v^T(k)\} = 0$

Stationary Kalman Filter V2 (Review)

The optimal state estimator is given by

$$\hat{x}^o(k+1) = \hat{A}\hat{x}^o(k) + \hat{B}u(k) + \hat{L}\tilde{y}^o(k)$$
$$\tilde{y}^o(k) = y(k) - \hat{C}\hat{x}^o(k)$$

where

$$\hat{L} = \hat{A}M\hat{C}^T[\hat{C}M\hat{C}^T + V]^{-1}$$
$$M = \hat{A}M\hat{A}^T + \hat{B}_w W\hat{B}_w^T - \hat{A}M\hat{C}^T[\hat{C}M\hat{C}^T + V]^{-1}\hat{C}M\hat{A}^T$$
$$\hat{A} - \hat{L}\hat{C} \text{ is Schur}$$

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$$\hat{A} - \hat{L}\hat{C} \text{ is Schur}$$

Also, the signal $\tilde{y}^o(k)$ is zero-mean, white, and has covariance $\hat{C}M\hat{C}^T + V$.

Alternate Model Form

Using the Kalman Filter V2, we can write

$$\hat{x}^o(k+1) = \hat{A}\hat{x}^o(k) + \hat{B}u(k) + \hat{L}\epsilon(k)$$
$$y(k) = \hat{C}\hat{x}^o(k) + \epsilon(k)$$

where $\epsilon(k) = \tilde{y}^o(k)$.

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where $\epsilon(k) = \tilde{y}^o(k)$.

As a transfer function, this is

$$\begin{aligned} Y(z) = {} & [\hat{C}(zI - \hat{A})^{-1}\hat{B}]U(z) \\ & + [1 + \hat{C}(zI - \hat{A})^{-1}\hat{L}]E(z) \end{aligned}$$

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Recall that $1 + \hat{C}(zI - \hat{A})^{-1}\hat{L} = \dfrac{\det[zI - (\hat{A} - \hat{L}\hat{C})]}{\det[zI - \hat{A}]}$

Alternate Transfer Function Model

From the previous slide, we have that

$$Y(z) = \frac{\bar{B}(z)}{\bar{A}(z)} U(z) + \frac{\bar{C}(z)}{\bar{A}(z)} E(z)$$

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$$\begin{aligned}
\bar{A}(z) &= z^n + a_1 z^{n-1} + \cdots + a_n &&= \det[zI - \hat{A}] \\
\bar{C}(z) &= z^n + c_1 z^{n-1} + \cdots + c_n &&= \det[zI - (\hat{A} - \hat{L}\hat{C})] \\
\bar{B}(z) &= b_0 z^m + \cdots + b_m
\end{aligned}$$

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\bar{B}(z) &= b_0 z^m + \cdots + b_m
\end{aligned}$$

Since $\hat{A} - \hat{L}\hat{C}$ is Schur, the polynomial $\bar{C}(z)$ is Schur

Polynomials in q^{-1}

We now define $\mathrm{d} = n - m$ and the polynomials

$$
\begin{aligned}
A(z^{-1}) &= z^{-n}\bar{A}(z) &&= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\
C(z^{-1}) &= z^{-n}\bar{C}(z) &&= 1 + c_1 z^{-1} + \cdots + c_n z^{-n} \\
B(z^{-1}) &= z^{-m}\bar{B}(z) &&= b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}
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\end{aligned}
$$

so that we can write the transfer function from the previous slide as

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Y(z) = \frac{z^{-\mathrm{d}}B(z^{-1})}{A(z^{-1})}U(z) + \frac{C(z^{-1})}{A(z^{-1})}E(z)
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so that we can write the transfer function from the previous slide as

$$Y(z) = \frac{z^{-\mathrm{d}} B(z^{-1})}{A(z^{-1})} U(z) + \frac{C(z^{-1})}{A(z^{-1})} E(z)$$

Note in particular that $C(z^{-1})$ is an anti-Schur polynomial of z^{-1}

ARMAX Plant Model

We have now transformed the original state space plant model to

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)$$

where $C(q^{-1})$ is an anti-Schur polynomial of q^{-1} and $\epsilon(k)$ is zero-mean white noise with covariance $\hat{C}M\hat{C}^T + V$

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This type of model is called an <u>ARMAX</u> model because it is an ARMA model with an eXogenous input.

Outline

Minimum Variance Regulator (MVR) Problem

Given the ARMAX model

$$A(q^{-1})y(k) = q^{-\mathrm{d}}B(q^{-1})u(k) + C(q^{-1})\epsilon(k)$$

where

- $C(q^{-1})$ is an anti-Schur polynomial of q^{-1}
- $B(q^{-1})$ has no zeros on the unit circle
- $\epsilon(k)$ is zero-mean white noise
- The plant has no unstable pole-zero cancelations, i.e. the polynomials $A(q^{-1})$ and $B(q^{-1})$ have no common zeros such that $|q^{-1}| < 1$

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find the stabilizing feedback control law that minimizes the output variance $E\{y^2(k)\}$

Outline

Introduction

MVR Problem Statement

MVR Solution

Proof, Special Case: $B(q^{-1})$ Anti-Schur

A-causal but BIBO Systems

Proof, General Case

Factorization of B and $\bar{B}$

In general, the polynomial $\bar{B}(q) = q^m B(q^{-1})$ has

- m_s zeros strictly inside the unit circle (stable plant zeros)
- m_u zeros strictly outside the unit circle (unstable plant zeros)

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Perform the factorization

$$B(q^{-1}) = B^s(q^{-1}) B^u(q^{-1})$$

where

- $\bar{B}^s(q) = q^{m_s} B^s(q^{-1})$ has its zeros inside the unit circle
  (These are the stable plant zeros)
- $\bar{B}^u(q) = q^{m_u} B^u(q^{-1})$ has its zeros outside the unit circle
  (These are the unstable plant zeros)
- $\bar{B}^u(0) = 1$

Minimum Variance Regulator (MVR) Solution

- The optimal control $u_*(k)$ is given by

$$B^s(q^{-1})R(q^{-1})u_*(k) = -S(q^{-1})y(k)$$

where $R(q^{-1})$ and $S(q^{-1})$ are found by solving the Diophantine equation

$$C(q^{-1})\bar{B}^u(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})$$

where

$$R(q^{-1}) = 1 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}$$
$$S(q^{-1}) = s_0 + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s}$$

and $n_r = m_u + \mathrm{d} - 1$ and $n_s = n - 1$

Minimum Variance Regulator (MVR) Solution

- The optimal cost is

$$E\{y^2(k)\} = E\{\epsilon_f^2(k)\}$$

where $\epsilon_f(k)$ is defined in terms of $\epsilon(k)$ by the ARMA model

$$\bar{B}^u(q^{-1})\epsilon_f(k) = R(q^{-1})\epsilon(k)$$

Constructing the MVR

1. Find $\hat{L}$ using a stationary Kalman filter design
2. Construct $C(q^{-1}) = q^{-n} \det[qI - (\hat{A} - \hat{L}\hat{C})]$
3. Factor $B(q^{-1}) = B^s(q^{-1})B^u(q^{-1})$ as described previously (don't forget that $\bar{B}^u(0) = 1$)
4. Solve the Diophantine equation

$$C(q^{-1})\bar{B}^u(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})$$

5. Form the optimal controller

$$B^s(q^{-1})R(q^{-1})u_*(k) = -S(q^{-1})y(k)$$

Solution Comments

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$\Rightarrow \quad \epsilon_f(k) = \dfrac{R(q^{-1})}{\bar{B}^u(q^{-1})}\epsilon(k)$ has bounded covariance

Special Case: $B(q^{-1})$ is anti-Schur

When $B(q^{-1})$ is anti-Schur, we have

- $B^s(q^{-1}) = B(q^{-1})$
- $B^u(q^{-1}) = \bar{B}^u(q) = \bar{B}^u(q^{-1}) = 1$

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- Expressing $R(q^{-1}) = 1 + r_1 q^{-1} + \cdots + r_{n_r} q^{-n_r}$, the optimal cost is

$$
\begin{aligned}
E\{y^2(k)\} &= E\{[R(q^{-1})\epsilon(k)]^2\} \\
&= E\{[\epsilon(k) + r_1\epsilon(k-1) + \cdots + r_{n_r}\epsilon(k-n_r)]^2\} \\
&= E\{\epsilon^2(k)\} + r_1^2 E\{\epsilon^2(k-1)\} + \cdots + r_{n_r}^2 E\{\epsilon^2(k-n_r)\} \\
&= (1 + r_1^2 + \cdots + r_{n_r}^2) E\{\epsilon^2(k)\}
\end{aligned}
$$

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Therefore

$$\boxed{E\{y^2(k)\} = (1 + r_1^2 + \cdots + r_{n_r}^2)(\hat{C} M \hat{C}^T + V)}$$

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Comments on the notation in this proof:

- Capital letters always denote polynomials; lower case letters denote sequences (except d and q)
- Dependency of polynomials on q^{-1} will be omitted e.g. $\bar{B}^u$ will refer to $\bar{B}^u(q^{-1})$
- Dependency of sequences on k will be omitted e.g. y will refer to $y(k)$

Part 1: Rewrite Dynamics

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$$R[q^{-\mathrm{d}}Bu + C\epsilon] = [C - q^{-\mathrm{d}}S]y$$

$$\Rightarrow \quad Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

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From the previous slide:

$$Cy - q^{-\mathrm{d}}(Sy + BRu) - CR\epsilon = 0$$

- Define $z(k)$ in terms of $y(k)$ and $u(k)$ using

$$Cz = Sy + BRu$$

(note that we are not necessarily using the optimal control)

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$$\boxed{y(k) = z(k - \mathrm{d}) + \epsilon_f(k)}$$

Part 2: $E\{z(k-\mathrm{d})\epsilon_f(k)\} = 0$

- Since $\epsilon(k) = y(k) - E\{y(k)|y(k-1), y(k-2), \ldots\}$, we use least squares property 1 to see that

$$E\{y(k-\ell)\epsilon(k+p)\}, \qquad \forall \ell > 0, p \geq 0$$

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- $\epsilon_f(k+\mathrm{d}-1) = \epsilon(k+\mathrm{d}-1) + r_1\epsilon(k+\mathrm{d}-2) + \cdots + r_{\mathrm{d}-1}\epsilon(k)$

$$\Rightarrow \quad E\{y(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

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$$\Rightarrow \quad E\{y(k-\ell)\epsilon_f(k+\mathrm{d}-1)\} = 0 \qquad \forall \ell > 0$$

- Since $u(k)$ is a function of $y(k), y(k-1), \ldots$

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- Since $z(k)$ is a function of $y(k), y(k-1), \ldots$ and $u(k), u(k-1), \ldots$

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- Choosing $\ell = 1$ completes part 2

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- If we can make $E\{z^2\} = 0$, the control must be optimal

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- Also note that $E\{y^2\} = E\{\epsilon_f^2\}$, provided that the closed-loop system is stable

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & BR \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

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Since $C(q^{-1})B(q^{-1})$ is an anti-Schur polynomial of q^{-1}, the closed-loop system is stable $\blacksquare$

Outline

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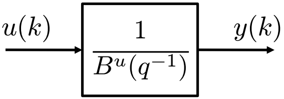

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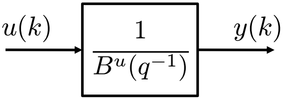

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We are considering the AR model $B^u(q^{-1})y(k) = u(k)$ where

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\Rightarrow b_{m_u}^u y(k) &= q^{m_u} u(k) - [q^{m_u} + b_1^u q^{m_u-1} + \cdots + b_{m_u-1}^u q]y(k) \\
\Rightarrow y(k) &= \frac{1}{b_{m_u}}[u(k+m_u) - y(k+m_u) - b_1^u y(k+m_u-1) \\
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A-causal but BIBO All-Pass Filter

Let $w(k)$ be the output of the a-causal, but BIBO ARMAX model

$$B^u(q^{-1})w(k) = \bar{B}^u(q^{-1})y(k)$$

This corresponds to the block diagram

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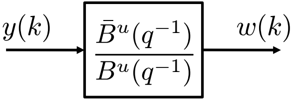

Claim:

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Proof:

$$\bar{B}^u(q) = q^{m_u} B^u(q^{-1}) \quad \Rightarrow \quad \bar{B}^u(q^{-1}) = q^{-m_u} B^u(q)$$

$$\Rightarrow \quad |\bar{B}^u(e^{-j\omega})| = |e^{-j\omega m_u} B^u(e^{j\omega})| = |B^u(e^{j\omega})| = |B^u(e^{-j\omega})| \qquad \blacksquare$$

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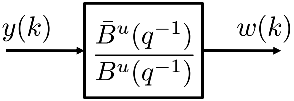

The power spectral density of $w(k)$ is

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The proof will be done in 4 parts:

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Comments on the notation in this proof:

- Capital letters always denote polynomials; lower case letters denote sequences (except d and q)
- Dependency of polynomials on q^{-1} will be omitted
  e.g. $\bar{B}^u$ will refer to $\bar{B}^u(q^{-1})$
- Dependency of sequences on k will be omitted
  e.g. y will refer to $y(k)$

Part 1: Rewrite Dynamics

The plant dynamics are

$$Ay = q^{-\mathrm{d}}Bu + C\epsilon$$

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Factoring B^u out of the term in parentheses yields

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- From the top equation,

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$$\Rightarrow CB^u (w - q^{-\mathrm{d}} z - \bar{\epsilon}_f) = 0$$

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- Defining $\epsilon_r = R\epsilon$, we have

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Part 3: Optimal Control

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- If we can make $E\{z^2\} = 0$, the control must be optimal

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- Also note that $E\{y^2\} = E\{\bar{\epsilon}_f^2\}$, provided that the closed-loop system is stable

Part 3: Optimal Control

- Provided that the closed-loop system is stable, we have $E\{y^2\} = E\{\bar{\epsilon}_f^2\}$ where $\bar{\epsilon}_f$ is generated by the BIBO a-causal ARMA model $B^u \bar{\epsilon}_f = R\epsilon$

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(Remember that $\bar{B}^u$ refers to $\bar{B}^u(q^{-1})$)

- To see this, note that since $\bar{\epsilon}_f$ is the output of the a-causal but BIBO ARMA model $B^u \bar{\epsilon}_f = \bar{B}^u \epsilon_f$ and the operator $\left(\dfrac{\bar{B}^u}{B^u}\right)$ is an a-causal all-pass filter, we have that $E\{\epsilon_f^2\} = E\{\bar{\epsilon}_f^2\}$

Part 4: Closed-loop stability

From the plant dynamics and feedback law, we have

$$\begin{bmatrix} A & -q^{-\mathrm{d}}B \\ S & B^{s}R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} y \\ u \end{bmatrix} = \frac{1}{B^sAR + q^{-\mathrm{d}}BS} \begin{bmatrix} B^sR & q^{-\mathrm{d}}B \\ -S & A \end{bmatrix} \begin{bmatrix} C\epsilon \\ 0 \end{bmatrix}$$

$$= \frac{1}{B^s(AR + q^{-\mathrm{d}}B^uS)} \begin{bmatrix} B^sRC\epsilon \\ -SC\epsilon \end{bmatrix} = \frac{1}{C\bar{B}^uB^s} \begin{bmatrix} B^sRC\epsilon \\ -SC\epsilon \end{bmatrix}$$

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Since $C(q^{-1})\bar{B}^u(q^{-1})B^s(q^{-1})$ is an anti-Schur polynomial of q^{-1}, the closed-loop system is stable ■