ME 233 Advanced Control II

Lecture 15

Deterministic Input/Output Approach to SISO Discrete-Time Systems

Repetitive Control

Deterministic SISO ARMA models

SISO ARMA model

$$A(q^{-1})\, y(k) = q^{-\mathrm{d}}\, B(q^{-1})\, [\, u(k) + d(k)]$$

Where all inputs and outputs are scalars:

- $u(k)$ control input
- $d(k)$ is a periodic disturbance of period N
- $y(k)$ output

Repetitive control assumptions

$$A(q^{-1})\, y(k) = q^{-\mathsf{d}}\, B(q^{-1})\, [\, u(k) + d(k)]$$

Both the disturbance and the reference model output are periodic sequences,

$$\left[1 - q^{-N}\right] d(k) \;\; = \;\; 0$$

$$\left[1 - q^{-N}\right] y_d(k) \;\; = \;\; 0$$

where N is a ***known*** and large number

Deterministic SISO ARMA models

$$A(q^{-1})\,y(k) = q^{-\mathrm{d}}\,B(q^{-1})\,[\,u(k) + d(k)]$$

Where the polynomials

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

are co-prime and d is the ***known*** pure time delay

Also, the polynomials $B(q^{-1})$ and $(1-q^{-N})$ are co-prime

Deterministic SISO ARMA models

The zero polynomial:

$$\bar{B}(q) = q^m \, B(q^{-1}) = 0$$

has

- m_u zeros that we **do not** want to cancel.
- m_s zeros inside the unit circle (asymptotically stable) that we **do** want to cancel.

$$B(q^{-1}) = B^s(q^{-1}) \, B^u(q^{-1})$$

$B^s(q^{-1})$ is anti-Schur

$\bar{B}^u(q) = q^{m_u} B^u(q^{-1})$ has the zeros (in *q*) that we **do not** want to cancel

Deterministic SISO ARMA models

The zero polynomial:

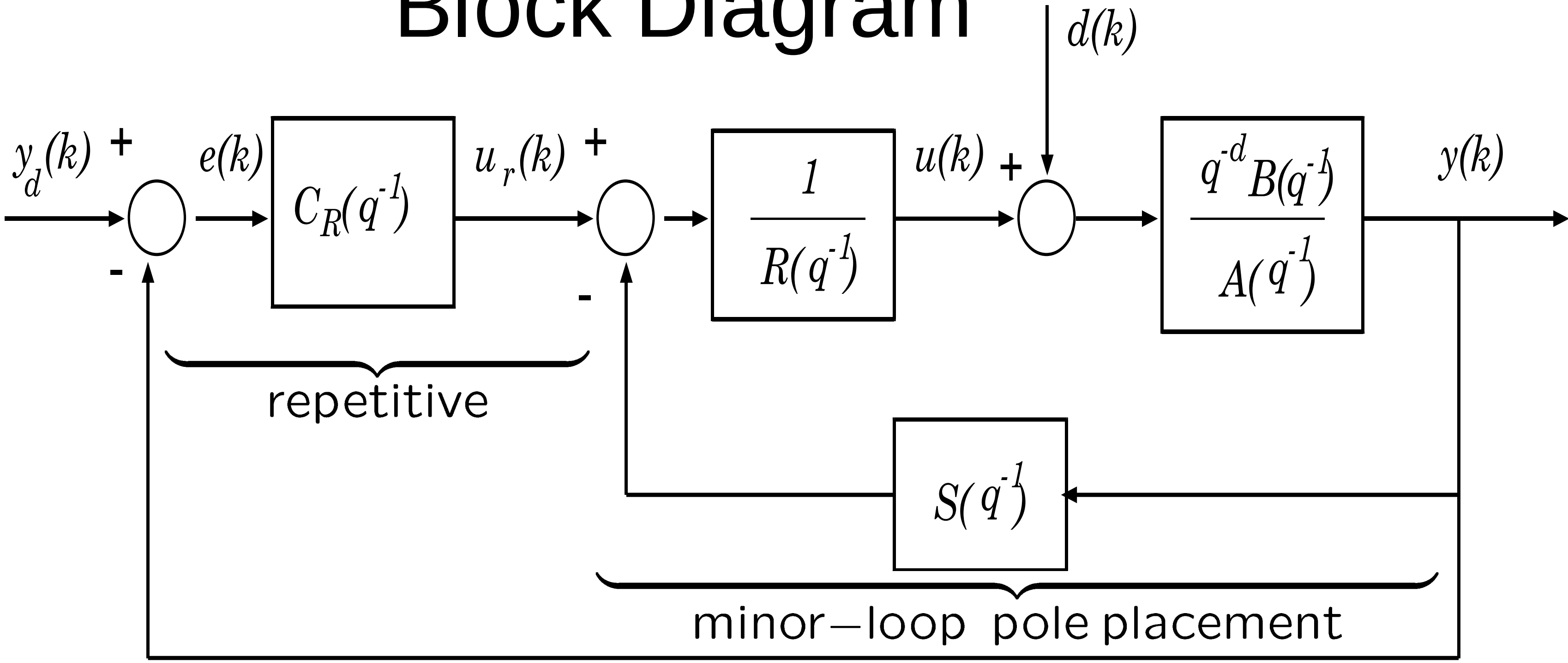
$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

Without loss of generality, we will assume that

$$B^s(q^{-1}) = 1 + \cdots + b^s_{m_s}\, q^{-m_s}$$

$$B^u(q^{-1}) = b_o + \cdots + b^u_{m_u}\, q^{-m_u}$$

Block Diagram



Control strategy: We design the controller in two stages

1. **Minor-loop pole placement:** Place minor-loop poles (these will be cancelled later)

2. **Repetitive compensator:** Reject periodic disturbance
   Follow periodic reference

Control Objectives

1. Minor-loop Pole Placement: The poles of the minor-loop system are placed at specific locations in the complex plane. **They will be cancelled later.**

- **Minor-loop pole polynomial:**

$$\boxed{A_c(q^{-1}) = B^s(q^{-1})\, A_c^{'}(q^{-1})}$$

Where:

- $B^s(q^{-1})$ cancelable plant zeros
- $A_c^{'}(q^{-1})$ anti-Schur polynomial chosen by the designer

$$A_c^{'}(q^{-1}) = 1 + a_{c1}^{'} q^{-1} + \cdots + a_{c n_c^{'}}^{'} q^{-n_c^{'}}$$

Control Objectives

2. Tracking: The output sequence $y(k)$ must asymptotically follow a ***reference*** sequence $y_d(k)$ which is periodic

$$\left[1 - q^{-N}\right] y_d(k) = 0$$

- **Error signal:**

$$\boxed{e(k) = y_d(k) - y(k)}$$

3. Disturbance rejection: The closed loop system must reject a class of deterministic disturbances which satisfy

$$\left[1 - q^{-N}\right] d(k) = 0$$

Step1: Minor-loop pole placement

Diophantine equation: Obtain polynomials $R(q^{-1}),\ S(q^{-1})$ that satisfy:

$$\boxed{\underbrace{A_c(q^{-1})}_{\text{Closed-loop poles}} = \underbrace{A(q^{-1})}_{\text{Plant poles}}\, \underline{R(q^{-1})} + q^{-\mathrm{d}} \underbrace{B(q^{-1})}_{\text{plant zeros}}\, \underline{S(q^{-1})}}$$

$$R(q^{-1}) = R'(q^{-1})\, \underline{B^s(q^{-1})}$$

$$A_c(q^{-1}) = \underline{B^s(q^{-1})}\, A'_c(q^{-1})$$

We will factor out the $B^s(q^{-1})$ *polynomial next*

The disturbance annihilating polynomial has not been included

Minor-loop pole placement

Diophantine equation: Obtain polynomials $\underline{R^{'}(q^{-1})}, \underline{S(q^{-1})}$ which satisfy:

$$\boxed{\underbrace{A_c^{'}(q^{-1})}_{\text{Closed-loop poles}} = \underbrace{A(q^{-1})}_{\text{Plant poles}} \underline{R^{'}(q^{-1})} + q^{-\mathrm{d}} \underbrace{B^u(q^{-1})}_{\text{Unstable plant zeros}} \underline{S(q^{-1})}}$$

Closed-loop poles

Plant poles

Unstable plant zeros

$$R(q^{-1}) = R^{'}(q^{-1})\, B^s(q^{-1})$$
$$A_c(q^{-1}) = B^s(q^{-1})\, A_c^{'}(q^{-1})$$

The disturbance annihilating polynomial has not been included

Diophantine equation

$$A_c^{'}(q^{-1}) = A(q^{-1})\,R^{'}(q^{-1}) + q^{-\mathsf{d}}B^u(q^{-1})\,S(q^{-1})$$

Solution:

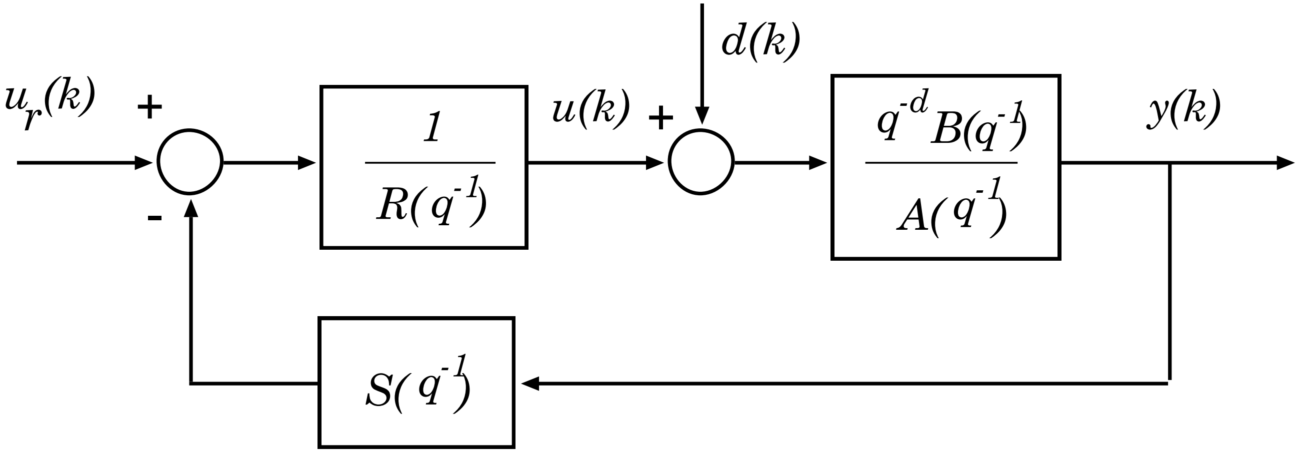
$$R^{'}(q^{-1}) \;=\; 1 + r_1^{'}q^{-1} + \cdots + r_{n_r^{'}}\,q^{-n_r^{'}}$$

$$S(q^{-1}) \;=\; s_o + s_1 q^{-1} + \cdots + s_{n_s} q^{-n_s}$$

$$n_r^{'} = \mathsf{d} + m_u - 1$$

$$n_s = \max\{\,n - 1\,,\,n_c^{'} - \mathsf{d} - m_u,\}$$

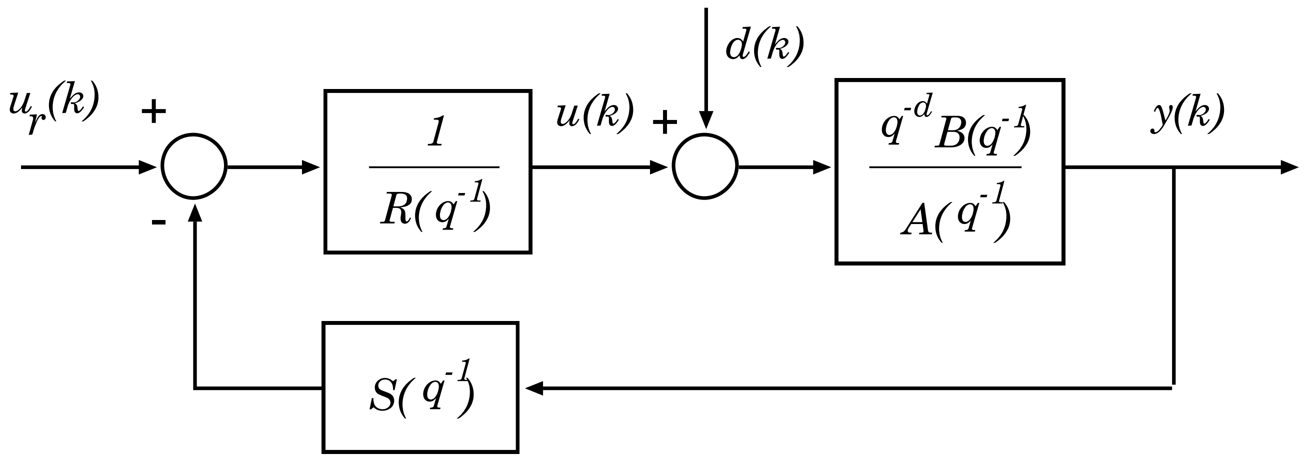
Minor-loop pole placement



$$u(k) = \frac{1}{R(q^{-1})}\left[u_r(k) - S(q^{-1})y(k)\right]$$

Minor-loop pole placement

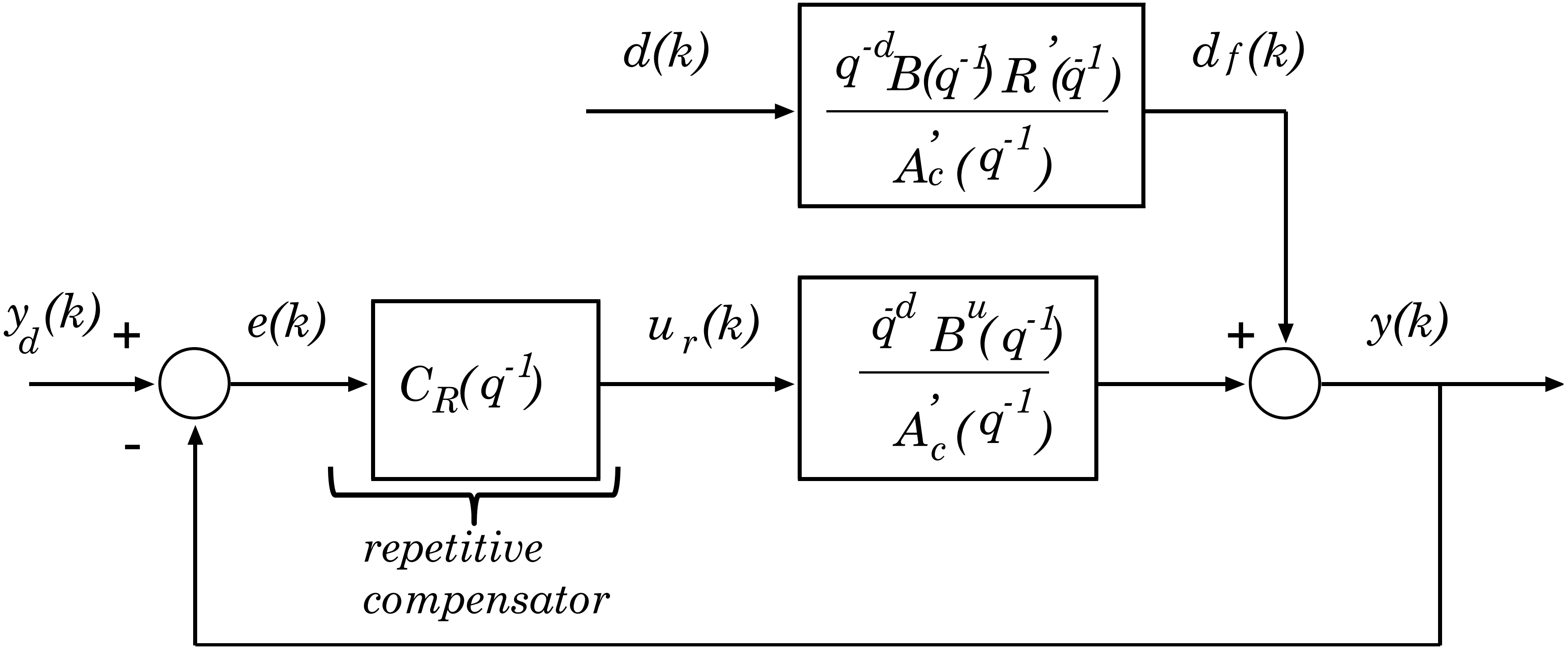
Closed-loop dynamics



$$y(k) = \frac{q^{-\mathsf{d}} B^u(q^{-1})}{A_c'(q^{-1})} u_r(k) + \underbrace{\frac{q^{-\mathsf{d}} B(q^{-1}) R'(q^{-1})}{A_c'(q^{-1})} d(k)}_{d_f(k)}$$

filtered repetitive disturbance

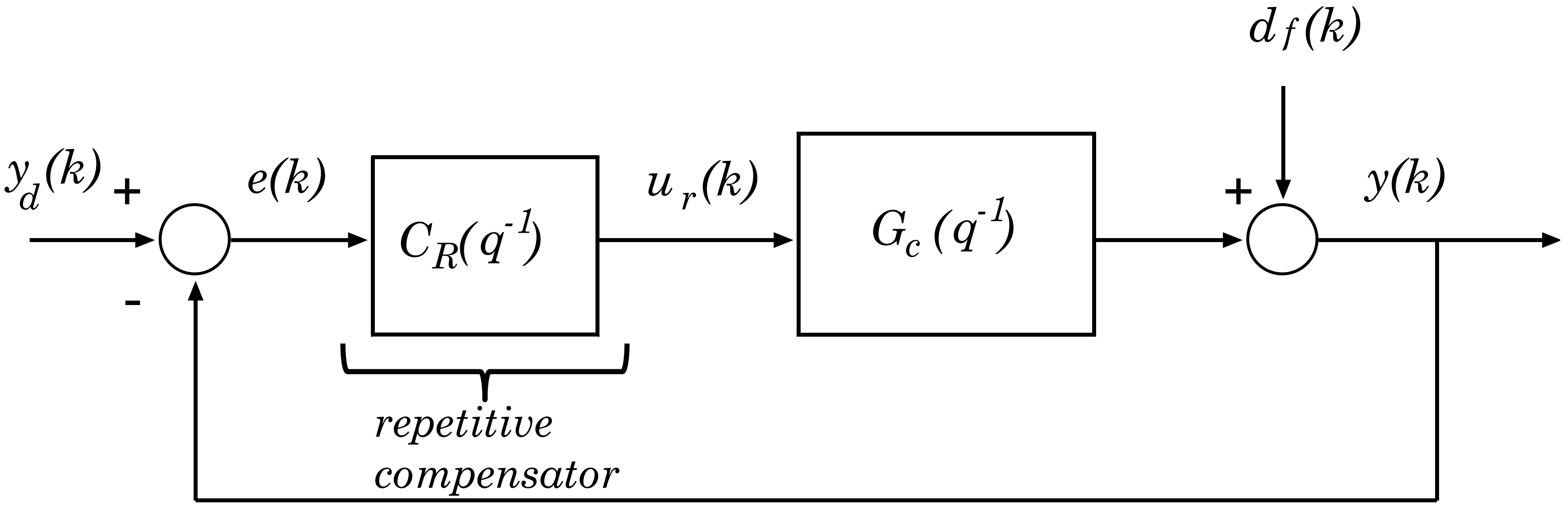
Equivalent Block Diagram



Notice that $d_f(k)$ is still a periodic disturbance

$$\left[1 - q^{-N}\right] \; y_d(k) \;=\; 0 \qquad\qquad \left[1 - q^{-N}\right] \; d_f(k) \;=\; 0$$

Equivalent Block Diagram

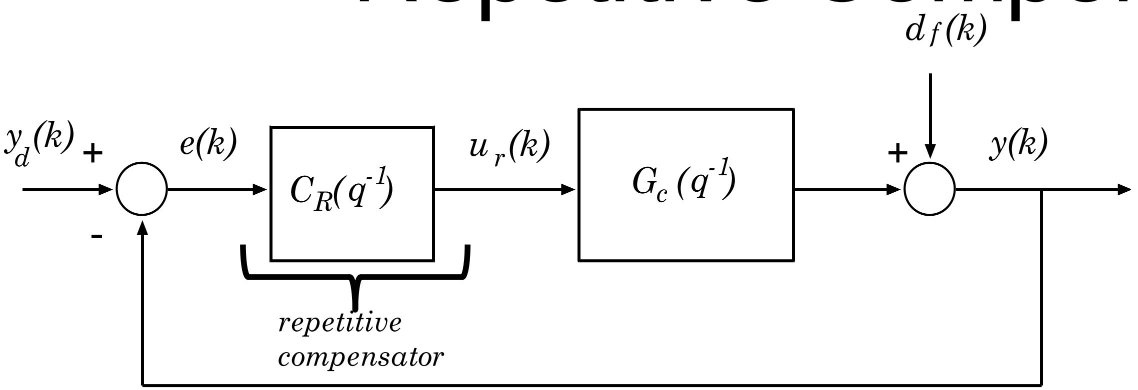


where

$$G_c(q^{-1}) = \frac{q^{-\mathsf{d}} B^u(q^{-1})}{A_c'(q^{-1})}$$

$$\left[1 - q^{-N}\right] y_d(k) = 0 \qquad \left[1 - q^{-N}\right] d_f(k) = 0$$

Repetitive Compensator



$$G_c(q^{-1}) = \frac{q^{-d} B^u(q^{-1})}{A'_c(q^{-1})}$$

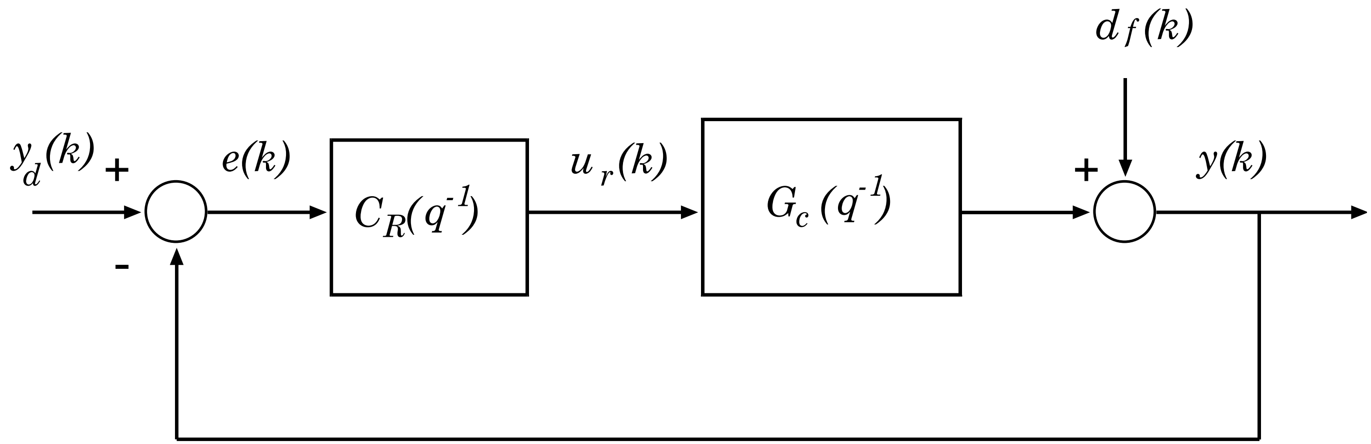
Repetitive compensator strategy:

1. Cancel stable poles and delay $A'_c(q^{-1}) \quad q^{-d}$
2. Zero-phase error compensation for $B^u(q^{-1})$
3. Include annihilating polynomial in the $1 - q^{-N}$ denominator

Not q^{-1}

$$C_R(q^{-1}) = \frac{k_r}{b}\left[\frac{q^{-N}}{1-q^{-N}}\right]\left[q^{d} A'_c(q^{-1})\, B^u(q)\right]$$

Repetitive Compensator



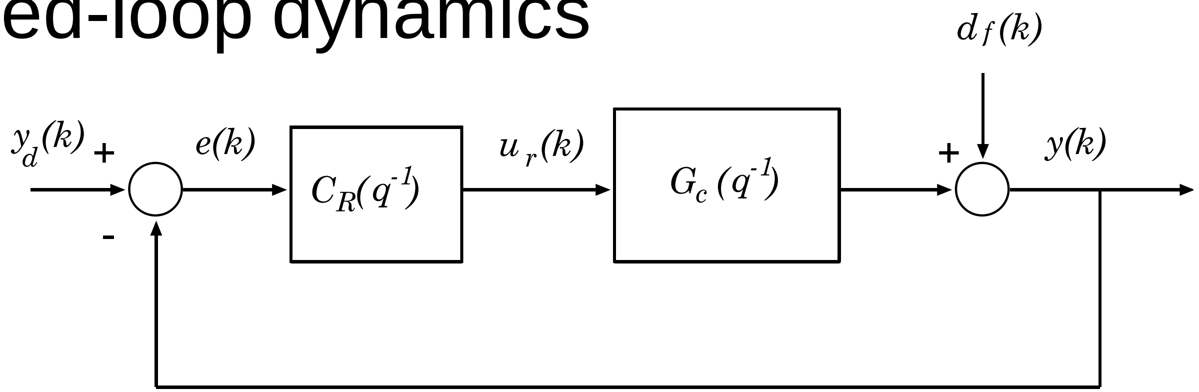
Repetitive compensator :

$$\boxed{C_R(q^{-1}) = \frac{k_r}{b}\left[\frac{q^{-N}}{1-q^{-N}}\right]\left[q^{\mathsf{d}} A_c'(q^{-1})\, B^u(q)\right]}$$

$(N \geq d + m_u)$ ⟵ *so that C_R is implementable*

Repetitive Compensator

- Closed-loop dynamics



$$e(k) = \frac{1}{1 + C_R(q^{-1})G_c(q^{-1})} \left[y_d(k) - d_f(k)\right]$$

$$G_c(q^{-1}) = \frac{q^{-\mathsf{d}} B^u(q^{-1})}{A_c'(q^{-1})}$$

$$C_R(q^{-1}) = \frac{k_r}{b} \left[\frac{q^{-N}}{1 - q^{-N}}\right] \left[q^{\mathsf{d}} A_c'(q^{-1})\, B^u(q)\right]$$

Repetitive Controller

Closed-loop dynamics: doing a bit of algebra, we obtain,

$$e(k) = \frac{q^N - 1}{\bar{A}_{cr}(q)} \left[y_d(k) - d_f(k) \right]$$

Where the closed-loop poles are the zeros of

$$\bar{A}_{cr}(q) = \left(q^N - 1 \right) + \frac{k_r}{b} B^u(q) \, B^u(q^{-1})$$

Repetitive Controller

since,

$$(q^N - 1)\left(y_d(k) - d_f(k)\right) = 0$$

we obtain

$$\boxed{\bar{A}_{cr}(q)e(k) = 0}$$

Where

$$\bar{A}_{cr}(q) = \left(q^N - 1\right) + \frac{k_r}{b}\, B^u(q)\, B^u(q^{-1})$$

Repetitive Controller

Theorem

The tracking error $e(k) \to 0$ if the gains $k_r,\ b$ are selected as follows:

1. $$b \geq \max_{\omega \in [0,\pi]} |B^u(e^{j\omega})|^2$$

2. $$0 < k_r < 2$$

Closed-loop poles for minimum phase zeros

Consider now the case when there are **no unstable zeros**,

e.g. $B^u(q^{-1}) = 1$

choose b = 1 so that $$\frac{B^u(q)\,B^u(q^{-1})}{b} = 1$$

Then the closed-loop poles are given by

$$\left(q^N - 1\right) + k_r = 0 \quad \rightarrow \quad q^N = 1 - k_r$$

Closed-loop poles for minimum phase zeros

For the case when the are no unstable zeros, the closed-loop poles are given by the roots of

$$q^N = 1 - k_r$$

When $0 < k_r < 2$, we have
N asymptotically stable closed-loop poles

<u>Case 1</u>: $0 < k_r \cdot 1$

$$\lambda_i = |1 - k_r|^{\frac{1}{N}} \exp\left\{j \frac{2\pi i}{N}\right\} \qquad i = 0, 1, \ldots, N-1$$

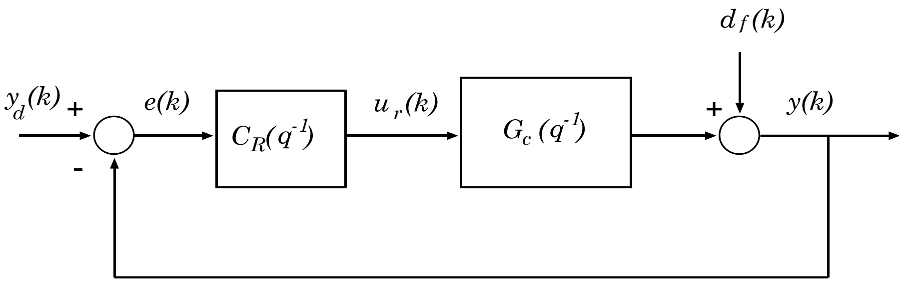
<u>Case 1</u>: $1 < k_r < 2$

$$\lambda_i = |1 - k_r|^{\frac{1}{N}} \exp\left\{j \frac{\pi(2i+1)}{N}\right\} \qquad i = 0, 1, \ldots, N-1$$

Repetitive control example

$$(B^u(q^{-1}) = b_o)$$
$$(d = 1)$$

Assume that $\boxed{N = 4}$



$$G_c(q^{-1}) = \frac{q^{-d}B^u(q^{-1})}{A_c'(q^{-1})} = \frac{b_o q^{-1}}{A_c'(q^{-1})}$$

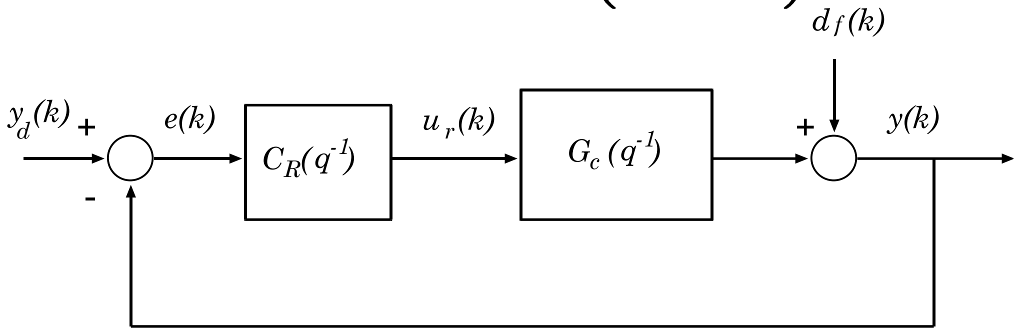
Choose $\quad b = b_o^2$

$$C_R(q^{-1}) = \frac{k_r}{b} q^{-(N-d)} \frac{A_c'(q^{-1})B^u(q)}{1 - q^{-N}} = \frac{k_r}{b_o} q^{-3} \frac{A_c'(q^{-1})}{1 - q^{-4}}$$

$$\Rightarrow \quad G_c(q^{-1})C_R(q^{-1}) \;=\; k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Repetitive control example

$$(B^u(q^{-1}) = b_o)$$
$$(d = 1)$$

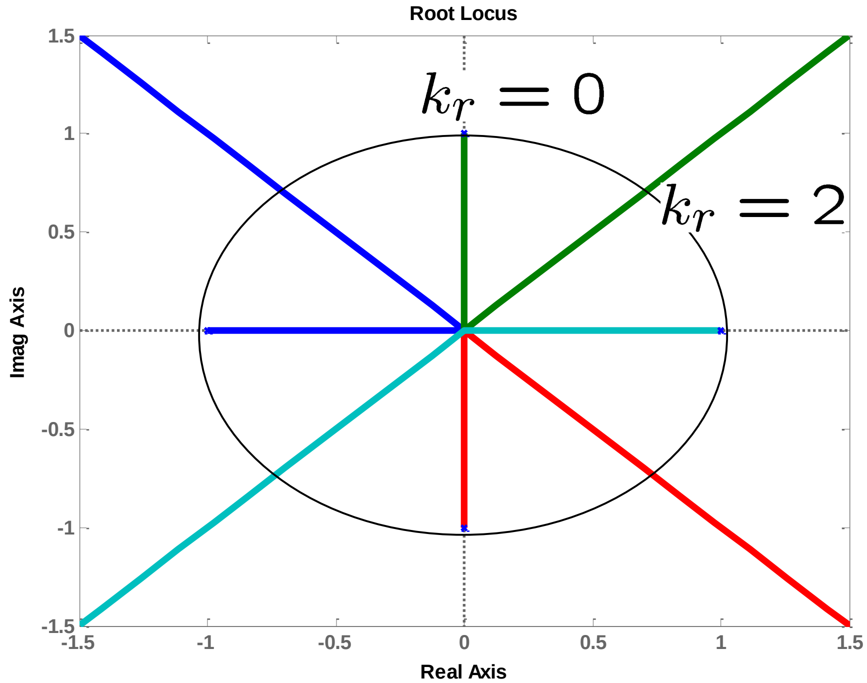


Open-loop TF

$$G_c(q^{-1})C_R(q^{-1}) \;=\; k_r\frac{q^{-4}}{1-q^{-4}} = k_r\frac{1}{q^4-1}$$

Closed-loop poles:

$$\boxed{1 + k_r\frac{1}{z^4-1} = 0}$$



Closed-loop poles for non-minimum phase zeros

Now consider the general case, i.e. there are unstable zeros

Assume that we have chosen *b* such that

$$\left| \frac{B^u(z)\, B^u(z^{-1})}{b} \right|_{z=e^{j\omega}} \leq 1, \qquad \forall \omega \in [0, \pi]$$

The closed-loop poles are the roots of

$$\left(q^N - 1\right) + k_r \, \frac{B^u(q)\, B^u(q^{-1})}{b} = 0$$

Closed-loop poles for non-minimum phase zeros

The closed-loop poles are the roots of

$$\left(z^N - 1\right) + k_r \, \frac{B^u(z)\, B^u(z^{-1})}{b} = 0$$

$$\Downarrow$$

$$1 - z^{-N} + \frac{\frac{k_r}{b} B^u(z) B^u(z^{-1})}{z^N} = 0$$

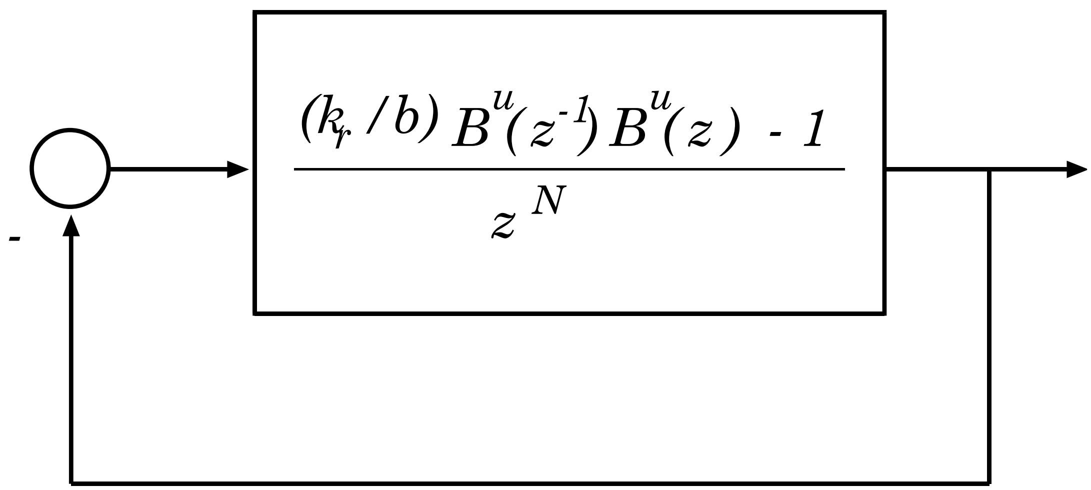
$$\Downarrow$$

$$1 + \frac{\frac{k_r}{b} B^u(z) B^u(z^{-1}) - 1}{z^N} = 0$$

Closed-loop poles for non-minimum phase zeros

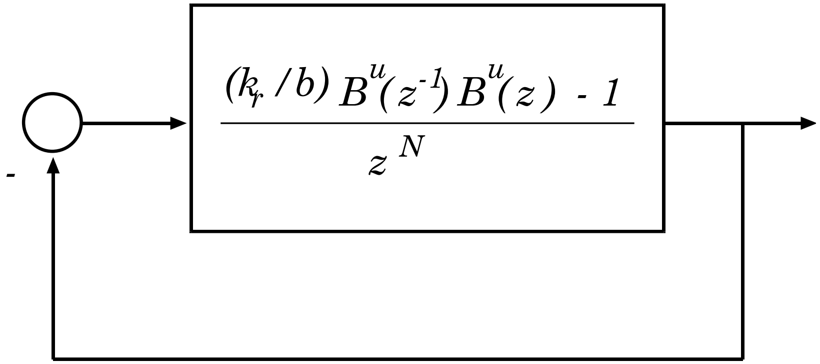
Therefore $\bar{A}_{cr}(z) = 0$ is equivalent to

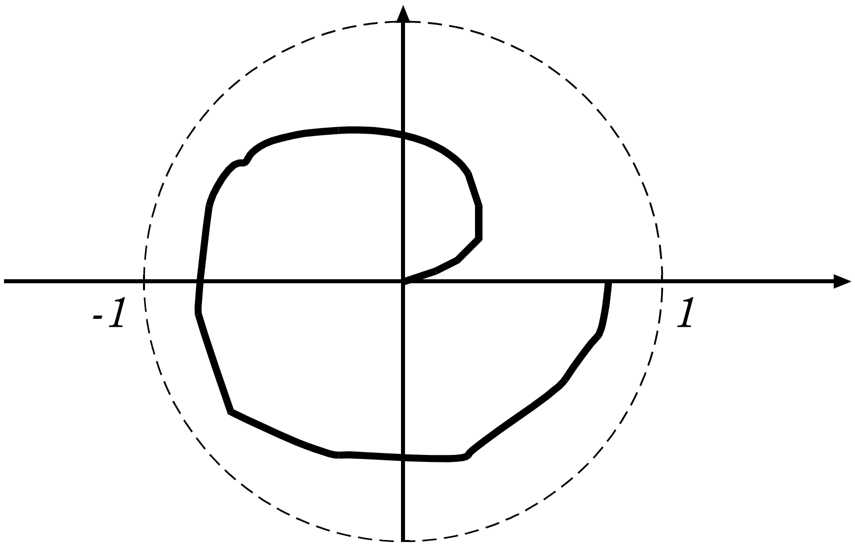
$$1 + \frac{\frac{k_r}{b} B^u(z)\, B^u(z^{-1}) - 1}{z^N} = 0$$



Closed-loop poles for non-minimum phase zeros

By Nyquist's theorem, the closed-loop system is asymptotically stable if there are no encirclements around –1.





This is guaranteed if the following hold for $\omega \in [0, \pi]$

$$\left| \frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| \leq 1 \qquad \frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \neq -1$$

Closed-loop poles for non-minimum phase zeros

Case 1: $B^u(e^{j\omega}) \neq 0$

We have $$0 < \frac{|B^u(e^{j\omega})|^2}{b} = \frac{B^u(e^{j\omega})\, B^u(e^{-j\omega})}{b} \leq 1$$

$$2 > k_r > 0 \qquad \Rightarrow \qquad 0 < \frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) < 2$$

$$\Rightarrow \qquad \left| \frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1 \right| < 1$$

$$|e^{j\omega N}| = 1 \qquad \Rightarrow \qquad \left| \frac{\frac{k_r}{b}\, B^u(e^{j\omega})\, B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| < 1$$

Closed-loop poles for non-minimum phase zeros

Case 2: $B^u(e^{j\omega}) = 0$

We have $\left| \dfrac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| = \left| \dfrac{-1}{e^{j\omega N}} \right| = 1$

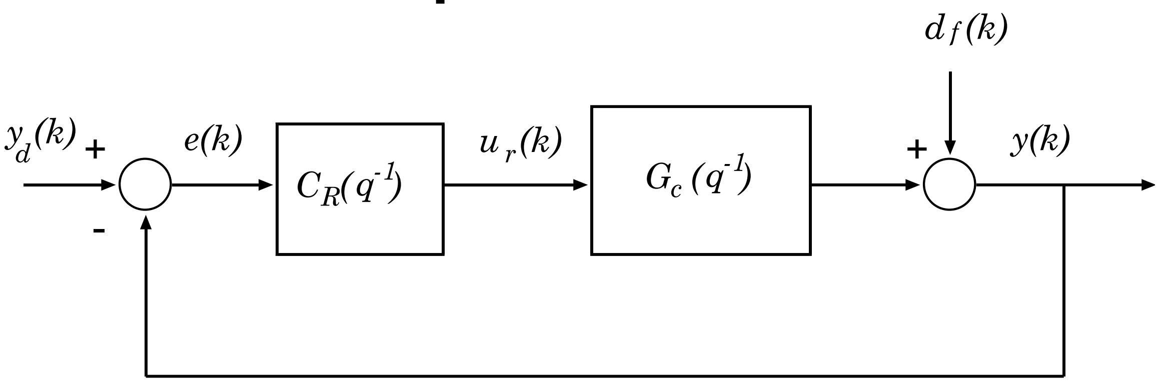
Since $B^u(q^{-1})$ and $1\text{-}q^{-N}$ are co-prime, we have that

$$1 - e^{j\omega N} \neq 0 \qquad \Rightarrow \qquad e^{j\omega N} \neq 1$$

$$\Rightarrow \qquad \frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} = \frac{-1}{e^{j\omega N}} \neq -1$$

$\Rightarrow$ Closed-loop stability

Repetitive Compensator



Repetitive compensator:

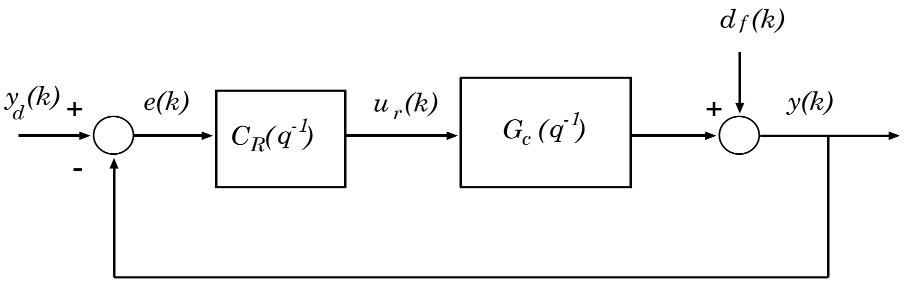
$$\boxed{C_R(q^{-1}) = \frac{k_r}{b}\underbrace{\left[\frac{q^{-N}}{1-q^{-N}}\right]}\left[q^{\mathsf{d}} A_c'(q^{-1})\, B^u(q)\right]}$$

The controller has N open-loop poles on the unit circle

Repetitive control example

$$(B^u(q^{-1}) = b_o)$$
$$(\mathsf{d} = 1)$$

Assume that $\boxed{N = 4}$



$$G_c(q^{-1}) = \frac{q^{-\mathsf{d}}B^u(q^{-1})}{A_c'(q^{-1})} = \frac{b_o q^{-1}}{A_c'(q^{-1})}$$

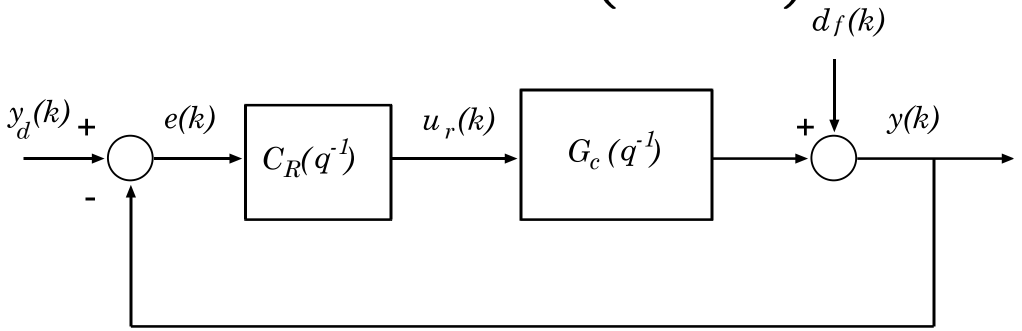
Choose $\quad b = b_o^2$

$$C_R(q^{-1}) = \frac{k_r}{b} q^{-(N-\mathsf{d})} \frac{A_c'(q^{-1})B^u(q)}{1 - q^{-N}} = \frac{k_r}{b_o} q^{-3} \frac{A_c'(q^{-1})}{1 - q^{-4}}$$

$$\Rightarrow \quad G_c(q^{-1})C_R(q^{-1}) = k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Repetitive control example

$$(B^u(q^{-1}) = b_o)$$
$$(d = 1)$$

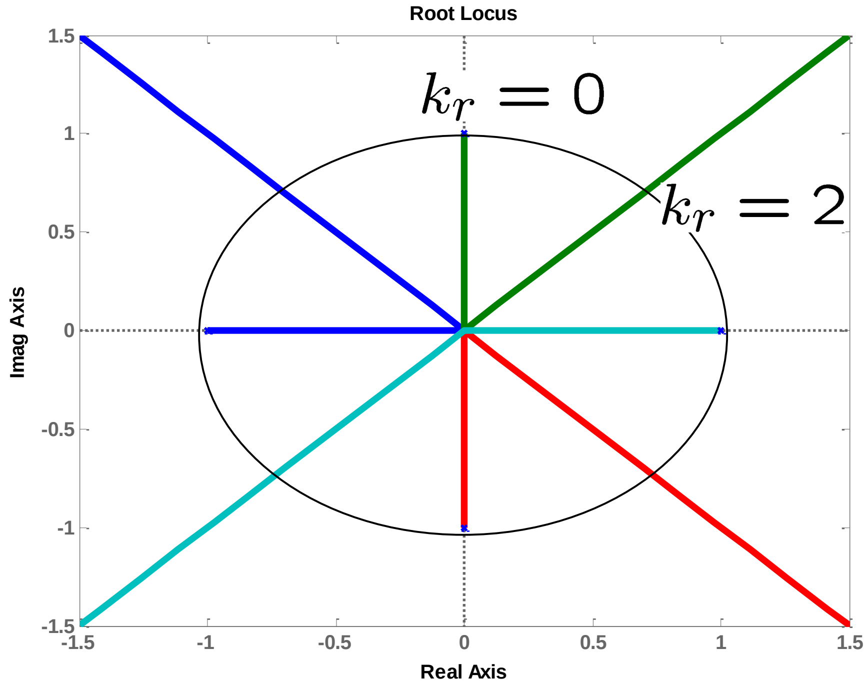


Open-loop TF

$$G_c(q^{-1})C_R(q^{-1}) = k_r \frac{q^{-4}}{1-q^{-4}} = k_r \frac{1}{q^4-1}$$

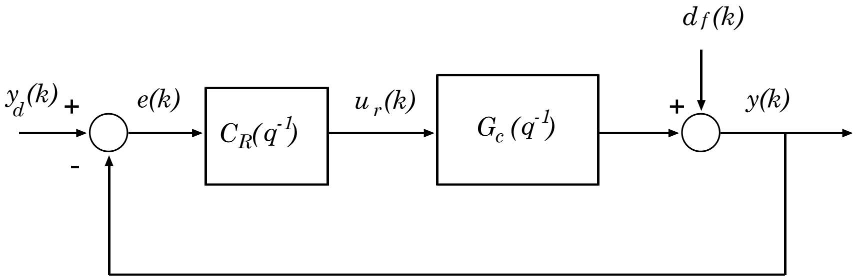
Closed-loop poles:

$$\boxed{1 + k_r \frac{1}{z^4 - 1} = 0}$$



Repetitive control, inexact cancellation

Assume that $\boxed{N = 4}$



Plant:

$$G_c(q^{-1}) = \frac{q^{-1}}{A_c'(q^{-1})} = \frac{q^{-1}}{\bar{A}_c'(q^{-1})} \frac{0.8\,q^{-1}}{1 - 0.2q^{-1}}$$

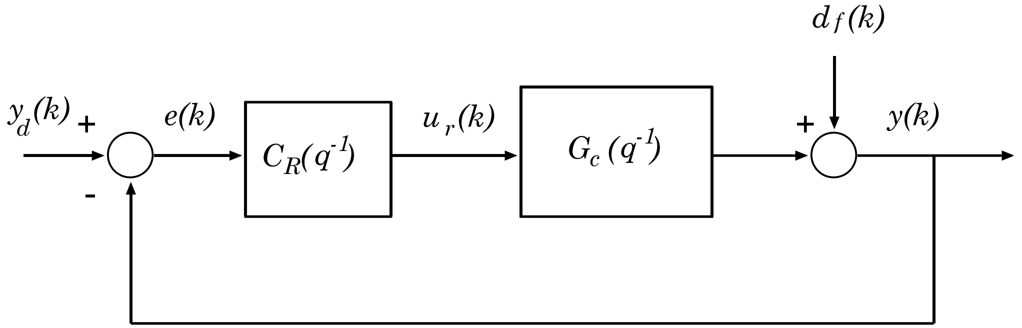
But, unmodeled dynamics are not cancelled

$$C_R(q^{-1}) = \frac{k_r}{b_o} q^{-3} \frac{\bar{A}_c'(q^{-1})}{1 - q^{-4}}$$

therefore,

$$G_c(q^{-1}) C_R(q^{-1}) = \frac{0.8\,k_r}{(q - 0.2)\,(q^4 - 1)}$$

Repetitive control, inexact cancellation

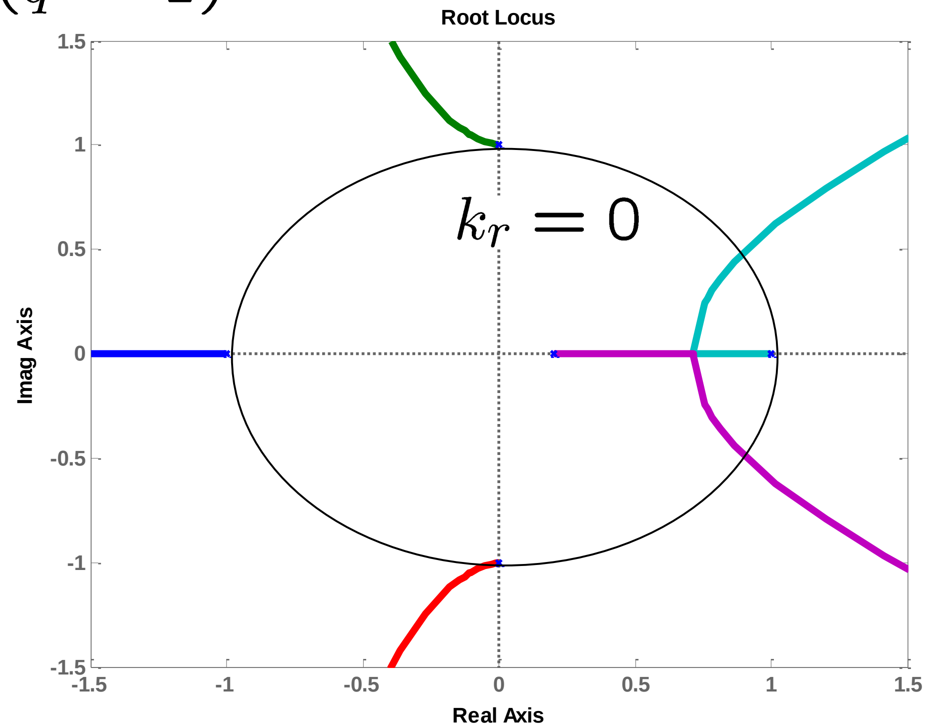


Open loop TF

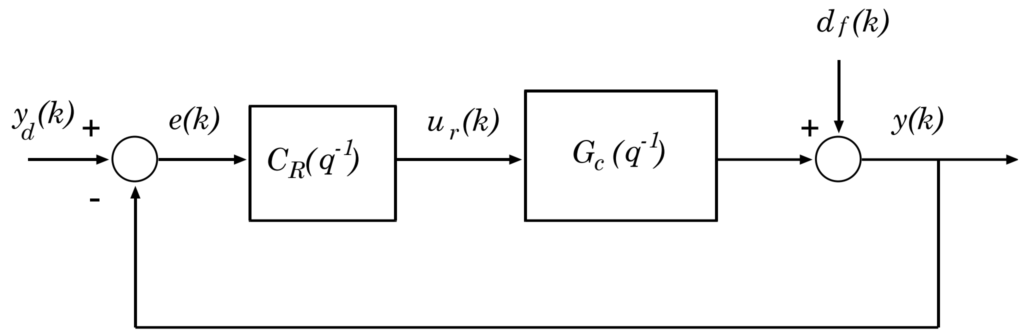
$$G_c(q^{-1})C_R(q^{-1}) = \frac{0.8\,k_r}{(q-0.2)\,(q^4-1)}$$

Closed-loop poles:

$$\boxed{1 + k_r\frac{0.8}{(z-0.2)(z^4-1)} = 0}$$



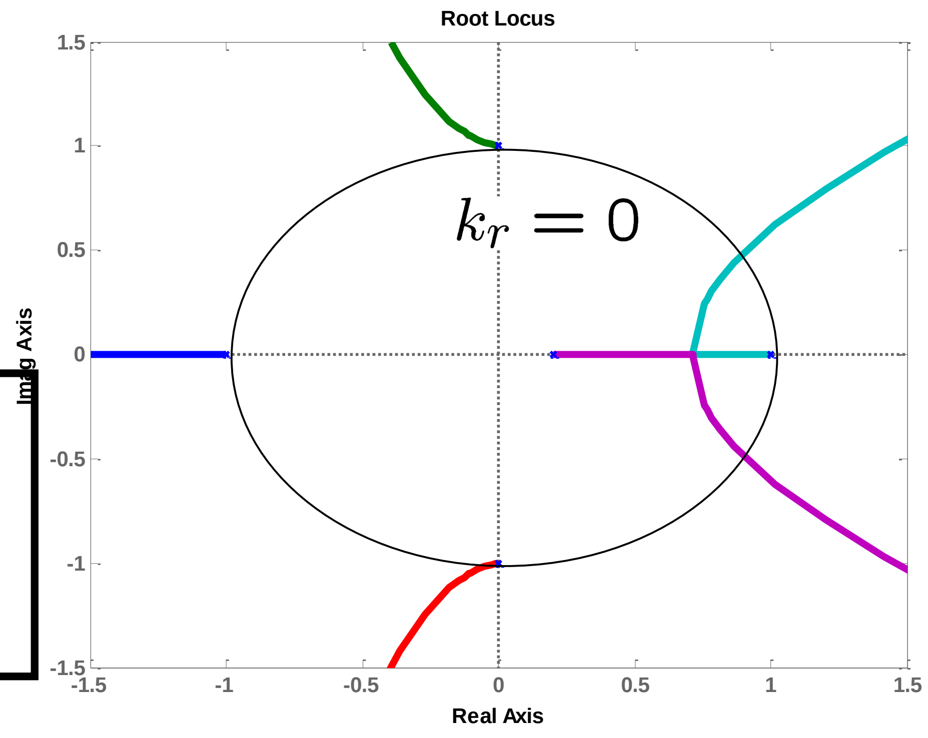
Repetitive control, inexact cancellation



Repetitive control is not robust to unmodeled dynamics

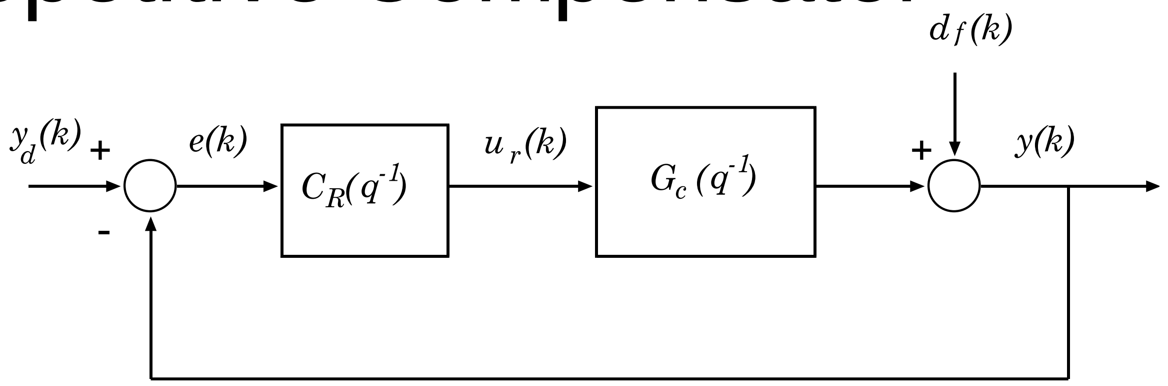
Closed-loop poles:

$$\boxed{1 + k_r \frac{0.8}{(z-0.2)(z^4-1)} = 0}$$



Robust Repetitive Compensator

Add Q-filter



$$C_R(q^{-1}) \;=\; \frac{k_r}{b}\, q^{-(N-d)}\, \frac{A_c'(q^{-1})\, B^u(q)}{1 - \underline{Q(q, q^{-1})}\, q^{-N}}$$

$Q(q, q^{-1})$ moving average filter with zero-phase shift characteristics

Controller’s N open-loop poles are no longer on the unit circle

Robust Repetitive Compensator

$Q(q, q^{-1})$ moving average filter with zero-phase shift characteristics

$$Q(q, q^{-1}) = \frac{\gamma_p q^p + \cdots \gamma_1 q + \gamma_o + \gamma_1 q^{-1} + \cdots \gamma_{p-1} q^{-(p-1)} + \gamma_p q^{-p}}{2\gamma_p + 2\gamma_{p-1} \cdots 2\gamma_1 + \gamma_o}$$

$N > p \qquad \gamma_o > \gamma_1 > \cdots > \gamma_p > 0$

$Q(q, q^{-1})$ has unit DC gain and gain decreases as frequency increases

Robust Repetitive Compensator

$$C_R(q^{-1}) = \frac{k_r}{b} q^{-(N-\mathrm{d})} \frac{A_c'(q^{-1}) B^u(q)}{1 - \underline{Q(q, q^{-1})} q^{-N}}$$

Notice that the disturbance $d(k)$ is no longer completely annihilated, since

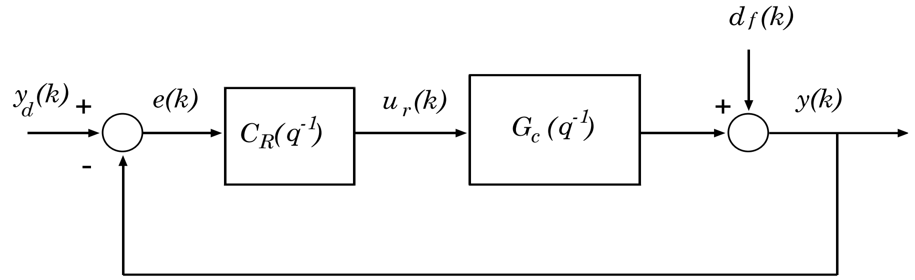
$$\left[1 - Q(q, q^{-1})\, q^{-N} \right] d(k) \neq 0$$

However, with a proper choice of Q filter,

$$\boxed{\left| \left[1 - Q(q, q^{-1})\, q^{-N} \right] d(k) \right| << |d(k)|}$$

Robust Rep. control, inexact cancellation

Assume that $\boxed{N = 4}$



Plant:

$$G_c(q^{-1}) = \frac{q^{-1}}{A_c'(q^{-1})} = \frac{q^{-1}}{\bar{A}_c'(q^{-1})} \frac{0.8\, q^{-1}}{1 - 0.2q^{-1}}$$

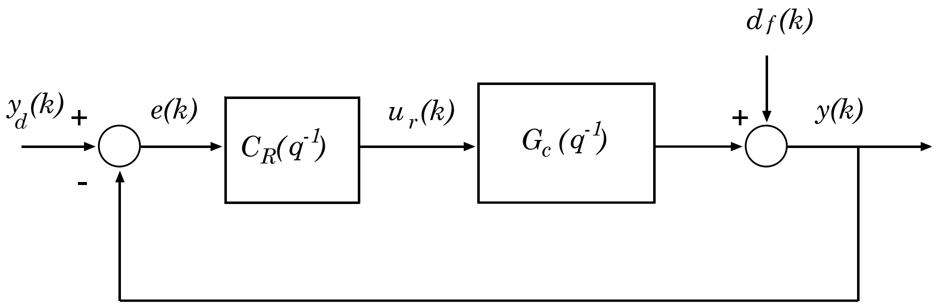
But, unmodeled dynamics are not cancelled

$$C_R(q^{-1}) = \frac{k_r}{b_o} q^{-3} \frac{\bar{A}_c'(q^{-1})}{1 - Q(q, q^{-1}) q^{-4}}$$

where,

$$Q(q, q^{-1}) = \frac{q + 4 + q^{-1}}{6}$$

Robust Rep. control, inexact cancellation



Closed-loop poles:

$$1 + k_r \frac{2.4\,z}{(z-0.2)(6\,z^5 - z^2 - 4\,z - 1)} = 0$$

Closed-loop system is asymptotically stable for a finite range of k_r