

ME 233 Advanced Control II

Lecture 15

Deterministic Input/Output Approach to SISO Discrete-Time Systems

Repetitive Control

Deterministic SISO ARMA models

SISO ARMA model

$$A(q^{-1}) y(k) = q^{-d} B(q^{-1}) [u(k) + d(k)]$$

Where all inputs and outputs are scalars:

- $u(k)$ control input
- $d(k)$ is a periodic disturbance of period N
- $y(k)$ output

Repetitive control assumptions

$$A(q^{-1}) y(k) = q^{-d} B(q^{-1}) [u(k) + d(k)]$$

Both the disturbance and the reference model output are periodic sequences,

$$\left[1 - q^{-N}\right] d(k) = 0$$

$$\left[1 - q^{-N}\right] y_d(k) = 0$$

where N is a **known** and large number

Deterministic SISO ARMA models

$$A(q^{-1}) y(k) = q^{-d} B(q^{-1}) [u(k) + d(k)]$$

Where the polynomials

$$A(q^{-1}) = 1 + a_1 q^{-1} + \dots + a_n q^{-n}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \dots + b_m q^{-m}$$

are co-prime and d is the **known** pure time delay

Also, the polynomials $B(q^{-1})$ and $(1-q^{-N})$ are co-prime

Deterministic SISO ARMA models

The zero polynomial:

$$\bar{B}(q) = q^m B(q^{-1}) = 0$$

has

- m_u zeros that we **do not** want to cancel.
- m_s zeros inside the unit circle (asymptotically stable) that we **do** want to cancel.

$$B(q^{-1}) = B^s(q^{-1}) B^u(q^{-1})$$

$$B^s(q^{-1})$$

is anti-Schur

$$\bar{B}^u(q) = q^{m_u} B^u(q^{-1})$$


has the zeros (in q) that we **do not** want to cancel

Deterministic SISO ARMA models

The zero polynomial:

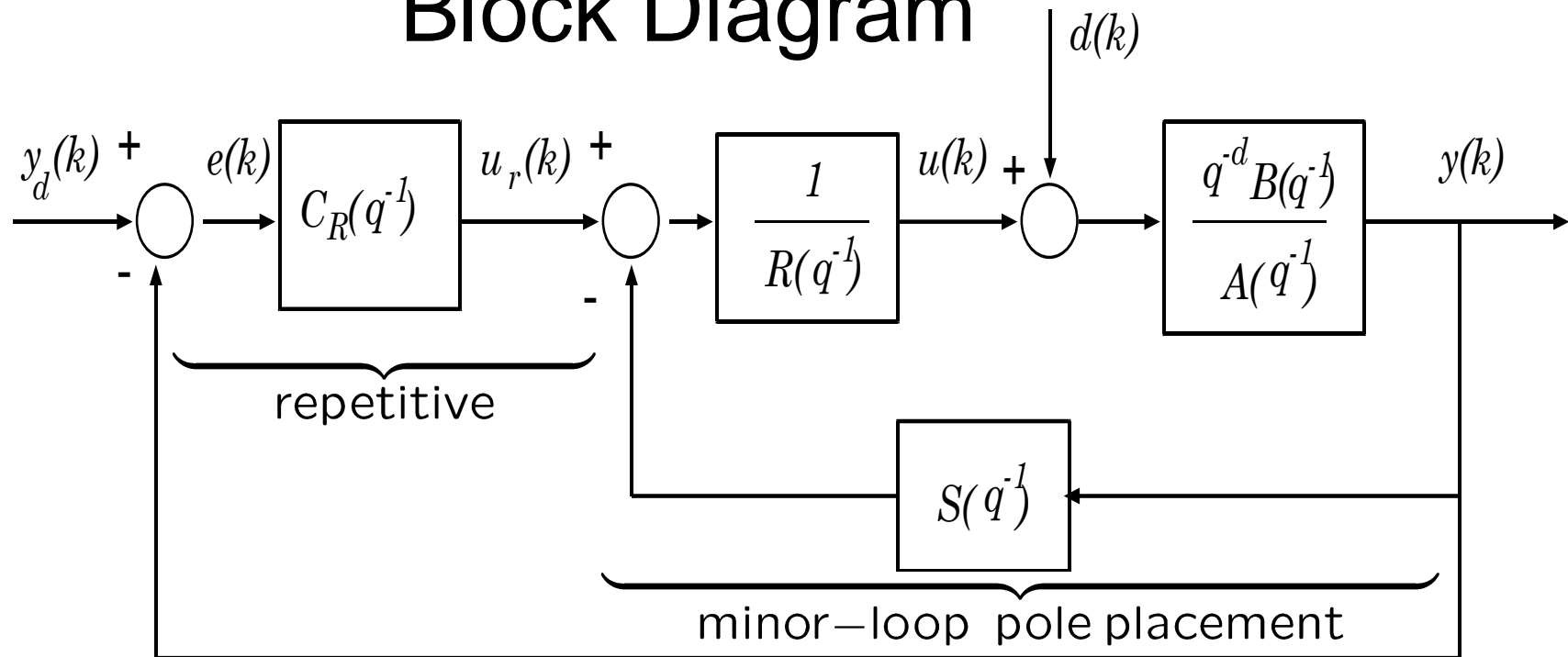
$$B(q^{-1}) = B^s(q^{-1}) B^u(q^{-1})$$

Without loss of generality, we will assume that

$$B^s(q^{-1}) = 1 + \cdots + b_{m_s}^s q^{-m_s}$$


$$B^u(q^{-1}) = b_o + \cdots + b_{m_u}^u q^{-m_u}$$

Block Diagram



Control strategy: We design the controller in two stages

- 1. Minor-loop pole placement:** Place minor-loop poles (these will be cancelled later)
- 2. Repetitive compensator:** Reject periodic disturbance
Follow periodic reference

Control Objectives

1. **Minor-loop Pole Placement:** The poles of the minor-loop system are placed at specific locations in the complex plane. **They will be cancelled later.**
 - **Minor-loop pole polynomial:**

$$A_c(q^{-1}) = B^s(q^{-1}) A'_c(q^{-1})$$

Where:

- $B^s(q^{-1})$ cancelable plant zeros
- $A'_c(q^{-1})$ anti-Schur polynomial chosen by the designer

$$A'_c(q^{-1}) = 1 + a'_{c1}q^{-1} + \dots + a'_{cn'_c}q^{-n'_c}$$

Control Objectives

2. **Tracking:** The output sequence $y(k)$ must asymptotically follow a **reference** sequence $y_d(k)$ which is periodic

$$\left[1 - q^{-N}\right] y_d(k) = 0$$

- **Error signal:**

$$e(k) = y_d(k) - y(k)$$

3. **Disturbance rejection:** The closed loop system must reject a class of deterministic disturbances which satisfy

$$\left[1 - q^{-N}\right] d(k) = 0$$

Step1: Minor-loop pole placement

Diophantine equation: Obtain polynomials $R(q^{-1})$, $S(q^{-1})$ that satisfy:

$$A_c(q^{-1}) = A(q^{-1}) \underline{R(q^{-1})} + q^{-d} B(q^{-1}) \underline{S(q^{-1})}$$

Closed-loop poles

Plant poles

plant zeros

$$R(q^{-1}) = R'(q^{-1}) \underline{B^s(q^{-1})}$$

$$A_c(q^{-1}) = \underline{B^s(q^{-1})} A'_c(q^{-1})$$

We will factor out the $B^s(q^{-1})$ polynomial next

The disturbance annihilating polynomial has not been included

Minor-loop pole placement

Diophantine equation: Obtain polynomials $R'(q^{-1})$, $S(q^{-1})$ which satisfy:

$$A'_c(q^{-1}) = A(q^{-1}) \underline{R'(q^{-1})} + q^{-d} B^u(q^{-1}) \underline{S(q^{-1})}$$

*Closed-loop
poles*

Plant poles

Unstable plant zeros

$$R(q^{-1}) = R'(q^{-1}) B^s(q^{-1})$$

$$A_c(q^{-1}) = B^s(q^{-1}) A'_c(q^{-1})$$

The disturbance annihilating polynomial has not been included

Diophantine equation

$$A'_c(q^{-1}) = A(q^{-1}) R'(q^{-1}) + q^{-d} B^u(q^{-1}) S(q^{-1})$$

Solution:

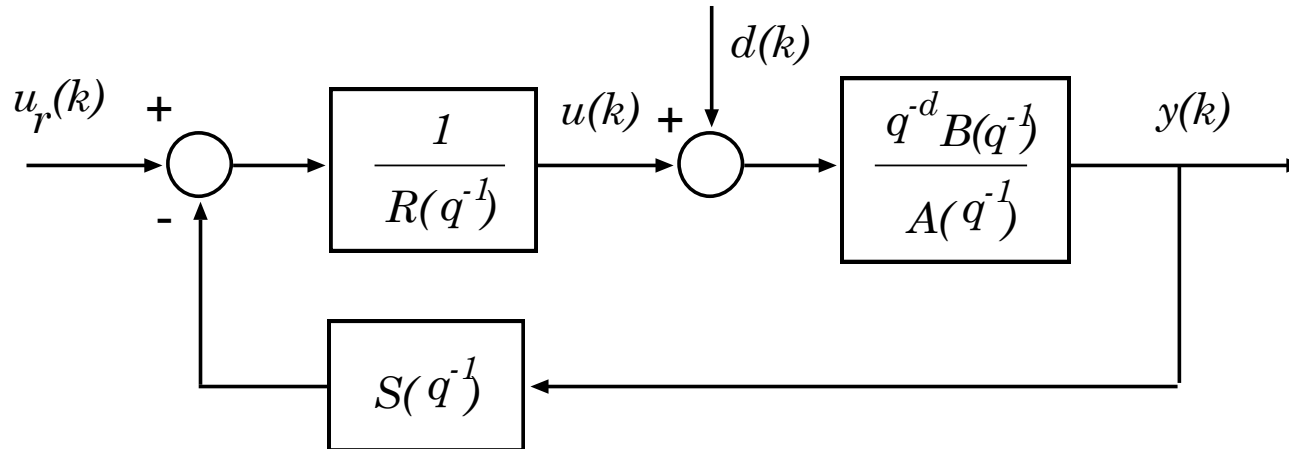
$$R'(q^{-1}) = 1 + r'_1 q^{-1} + \dots + r'_{n'_r} q^{-n'_r}$$

$$S(q^{-1}) = s_0 + s_1 q^{-1} + \dots + s_{n_s} q^{-n_s}$$

$$n'_r = d + m_u - 1$$

$$n_s = \max\{n - 1, n'_c - d - m_u, \}$$

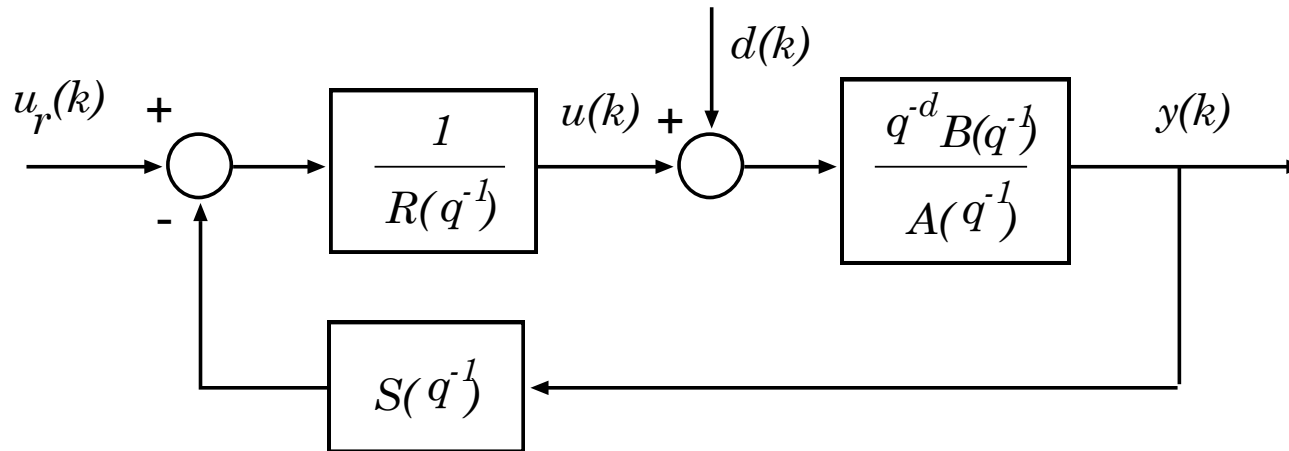
Minor-loop pole placement



$$u(k) = \frac{1}{R(q^{-1})} \left[u_r(k) - S(q^{-1})y(k) \right]$$

Minor-loop pole placement

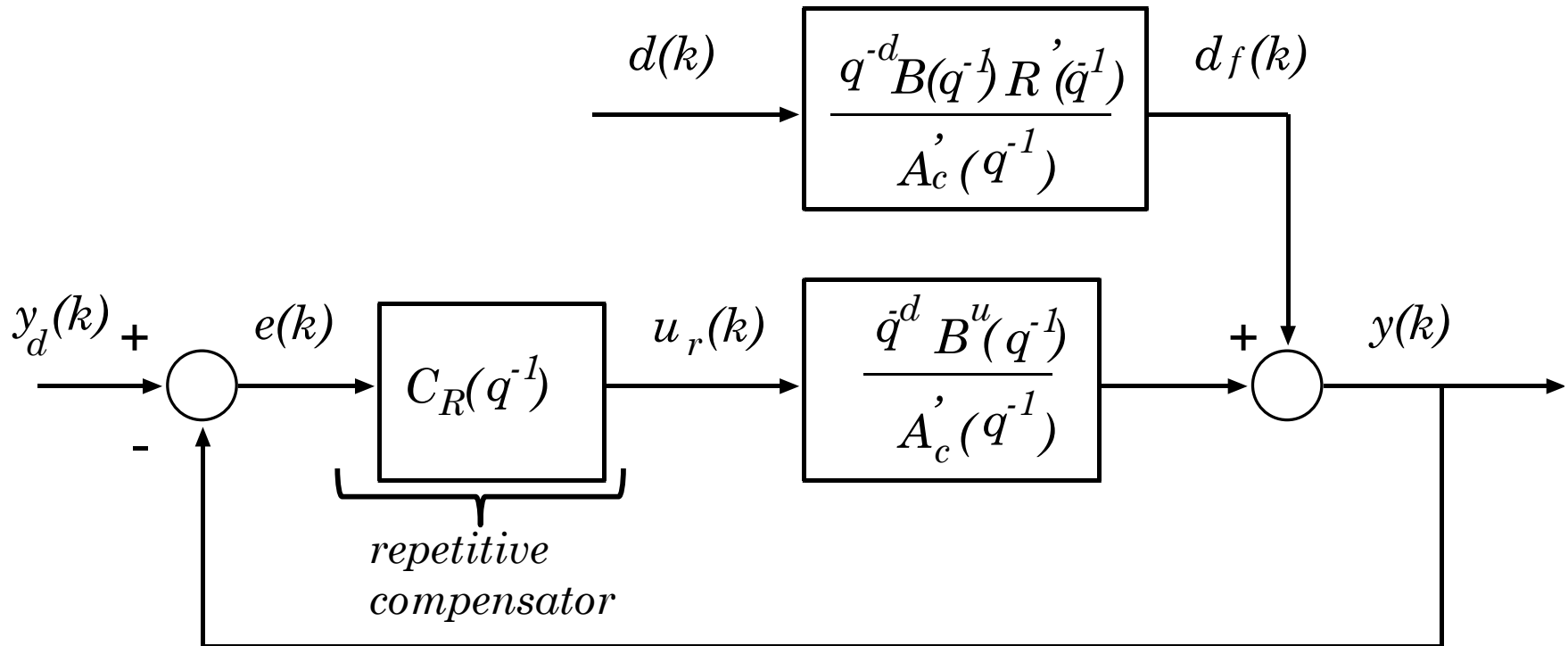
Closed-loop dynamics



$$y(k) = \frac{q^{-d} B(q^{-1})}{A'_c(q^{-1})} u_r(k) + \underbrace{\frac{q^{-d} B(q^{-1}) R'(q^{-1})}{A'_c(q^{-1})} d(k)}_{d_f(k)}$$

filtered repetitive disturbance

Equivalent Block Diagram

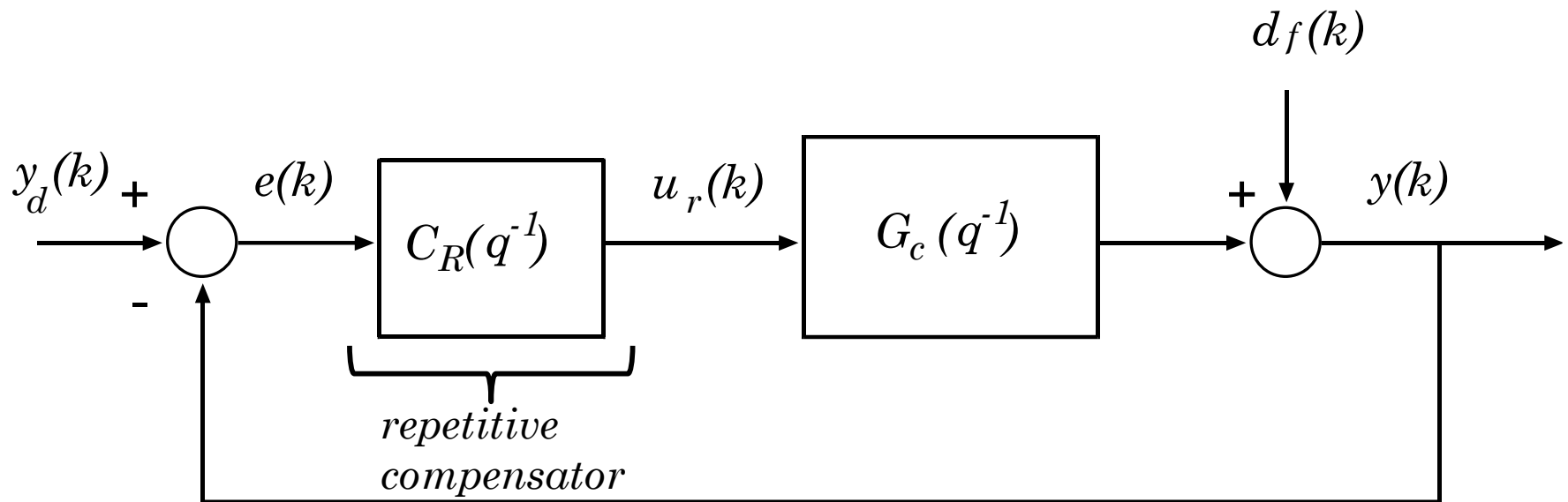


Notice that $d_f(k)$ is still a periodic disturbance

$$\left[1 - q^{-N}\right] y_d(k) = 0$$

$$\left[1 - q^{-N}\right] d_f(k) = 0$$

Equivalent Block Diagram



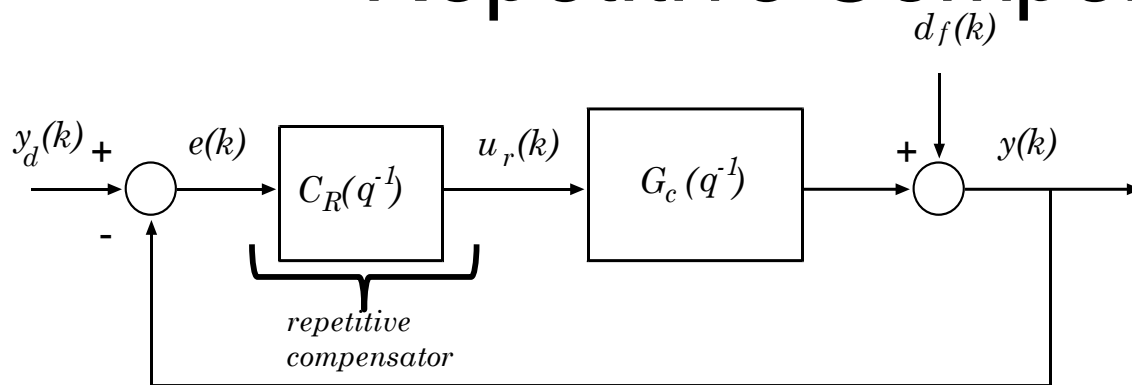
where

$$G_c(q^{-1}) = \frac{q^{-d} B^u(q^{-1})}{A'_c(q^{-1})}$$

$$[1 - q^{-N}] y_d(k) = 0$$

$$[1 - q^{-N}] d_f(k) = 0$$

Repetitive Compensator



$$G_c(q^{-1}) = \frac{q^{-d} B^u(q^{-1})}{A'_c(q^{-1})}$$

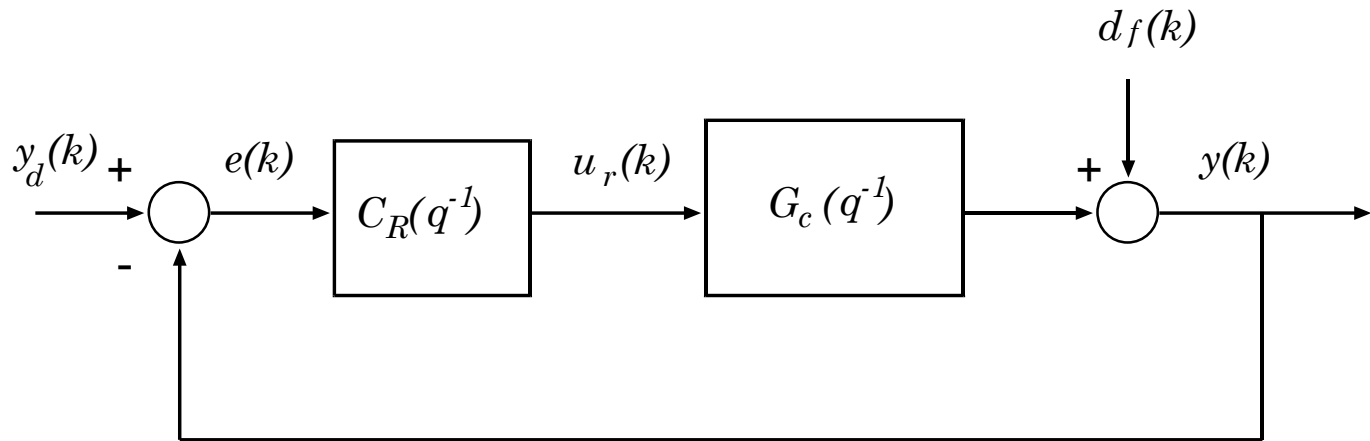
Repetitive compensator strategy:

1. Cancel stable poles and delay $A'_c(q^{-1}) q^{-d}$
2. Zero-phase error compensation for $B^u(q^{-1})$
3. Include annihilating polynomial in the denominator $1 - q^{-N}$

$$C_R(q^{-1}) = \frac{k_r}{b} \left[\frac{q^{-N}}{1 - q^{-N}} \right] \left[q^d A'_c(q^{-1}) B^u(q) \right]$$

Not q^{-1}

Repetitive Compensator



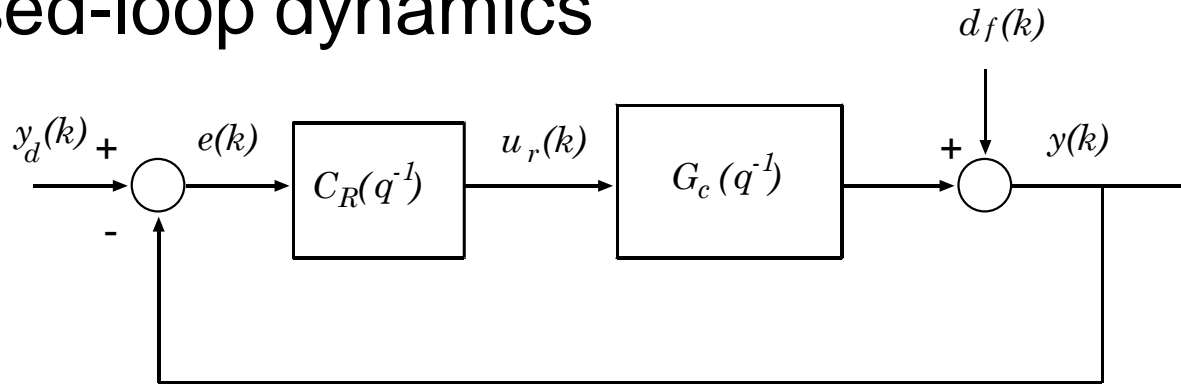
Repetitive compensator :

$$C_R(q^{-1}) = \frac{k_r}{b} \left[\frac{q^{-N}}{1 - q^{-N}} \right] \left[q^d A'_c(q^{-1}) B^u(q) \right]$$

$(N \geq d + m_u)$ ← so that C_R is implementable

Repetitive Compensator

- Closed-loop dynamics



$$e(k) = \frac{1}{1 + C_R(q^{-1})G_c(q^{-1})} [y_d(k) - d_f(k)]$$

$$G_c(q^{-1}) = \frac{q^{-d} B^u(q^{-1})}{A'_c(q^{-1})}$$

$$C_R(q^{-1}) = \frac{k_r}{b} \left[\frac{q^{-N}}{1 - q^{-N}} \right] [q^d A'_c(q^{-1}) B^u(q)]$$

Repetitive Controller

Closed-loop dynamics: doing a bit of algebra, we obtain,

$$e(k) = \frac{q^N - 1}{\bar{A}_{cr}(q)} [y_d(k) - d_f(k)]$$

Where the closed-loop poles are the zeros of

$$\bar{A}_{cr}(q) = (q^N - 1) + \frac{k_r}{b} B^u(q) B^u(q^{-1})$$

Repetitive Controller

since,

$$(q^N - 1) (y_d(k) - d_f(k)) = 0$$

we obtain

$$\bar{A}_{cr}(q)e(k) = 0$$

Where

$$\bar{A}_{cr}(q) = (q^N - 1) + \frac{k_r}{b} B^u(q) B^u(q^{-1})$$

Repetitive Controller

Theorem

The tracking error $e(k) \rightarrow 0$ if the gains k_r, b are selected as follows:

1. $b \geq \max_{\omega \in [0, \pi]} |B^u(e^{j\omega})|^2$
2. $0 < k_r < 2$

Closed-loop poles for minimum phase zeros

Consider now the case when there are **no unstable zeros**,

e.g. $B^u(q^{-1}) = 1$

choose $b = 1$ so that
$$\frac{B^u(q) B^u(q^{-1})}{b} = 1$$

Then the closed-loop poles are given by

$$(q^N - 1) + k_r = 0 \quad \rightarrow \quad q^N = 1 - k_r$$

Closed-loop poles for minimum phase zeros

For the case when there are no unstable zeros, the closed-loop poles are given by the roots of

$$q^N = 1 - k_r$$

When $0 < k_r < 2$, we have

N asymptotically stable closed-loop poles

Case 1: $0 < k_r < 1$

$$\lambda_i = |1 - k_r|^{\frac{1}{N}} \exp \left\{ j \frac{2\pi i}{N} \right\} \quad i = 0, 1, \dots, N - 1$$

Case 2: $1 < k_r < 2$

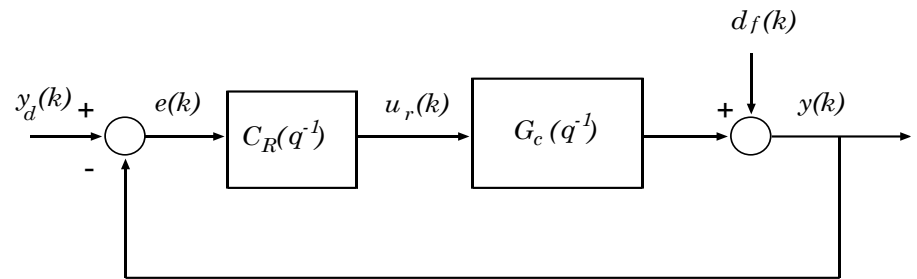
$$\lambda_i = |1 - k_r|^{\frac{1}{N}} \exp \left\{ j \frac{\pi(2i + 1)}{N} \right\} \quad i = 0, 1, \dots, N - 1$$

Repetitive control example

$$\begin{aligned} (B^u(q^{-1}) &= b_o) \\ (d &= 1) \end{aligned}$$

Assume that

$$N = 4$$



$$G_c(q^{-1}) = \frac{q^{-d} B^u(q^{-1})}{A'_c(q^{-1})} = \frac{b_o q^{-1}}{A'_c(q^{-1})}$$

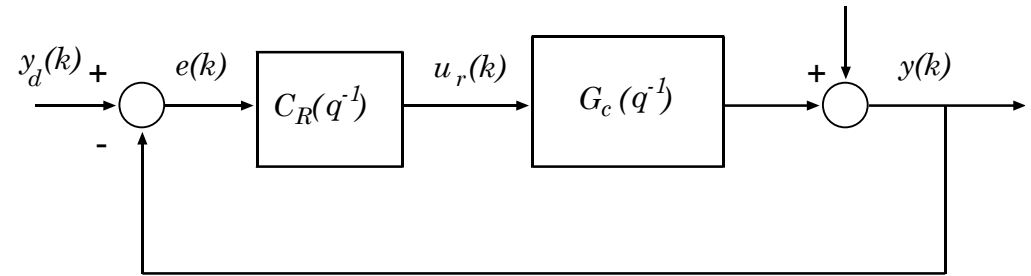
Choose $b = b_o^2$

$$C_R(q^{-1}) = \frac{k_r}{b} q^{-(N-d)} \frac{A'_c(q^{-1}) B^u(q)}{1 - q^{-N}} = \frac{k_r}{b_o} q^{-3} \frac{A'_c(q^{-1})}{1 - q^{-4}}$$

$$\rightarrow G_c(q^{-1}) C_R(q^{-1}) = k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Repetitive control example

$$\begin{aligned} (B^u(q^{-1}) &= b_o) \\ (d &= 1) \end{aligned}$$

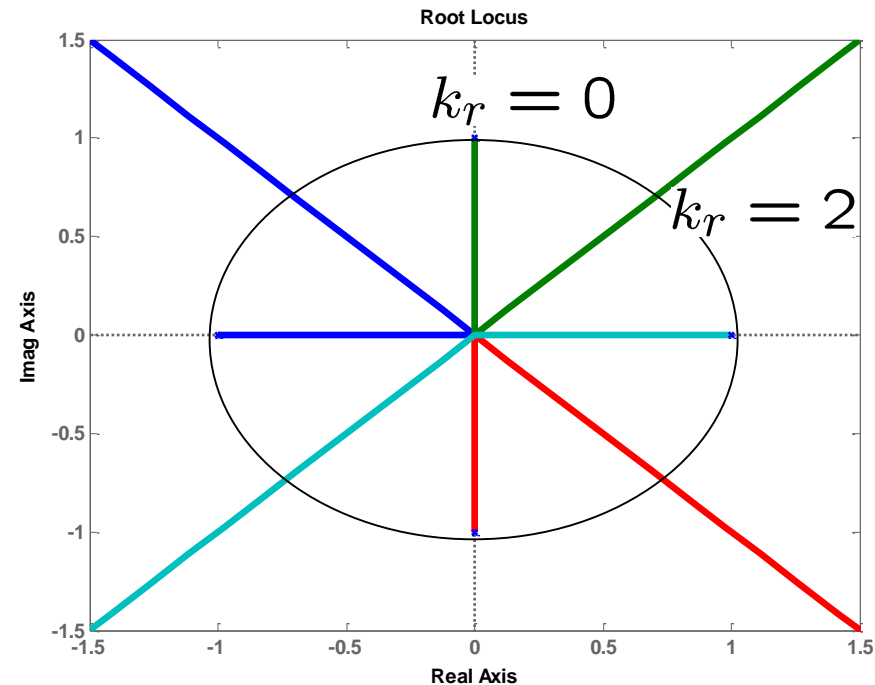


Open-loop TF

$$G_c(q^{-1})C_R(q^{-1}) = k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Closed-loop poles:

$$1 + k_r \frac{1}{z^4 - 1} = 0$$



Closed-loop poles for non-minimum phase zeros

Now consider the general case, i.e. there are unstable zeros

Assume that we have chosen b such that

$$\left| \frac{B^u(z) B^u(z^{-1})}{b} \right|_{z=e^{j\omega}} \leq 1, \quad \forall \omega \in [0, \pi]$$

The closed-loop poles are the roots of

$$(q^N - 1) + k_r \frac{B^u(q) B^u(q^{-1})}{b} = 0$$

Closed-loop poles for non-minimum phase zeros

The closed-loop poles are the roots of

$$(z^N - 1) + k_r \frac{B^u(z) B^u(z^{-1})}{b} = 0$$



$$1 - z^{-N} + \frac{\frac{k_r}{b} B^u(z) B^u(z^{-1})}{z^N} = 0$$

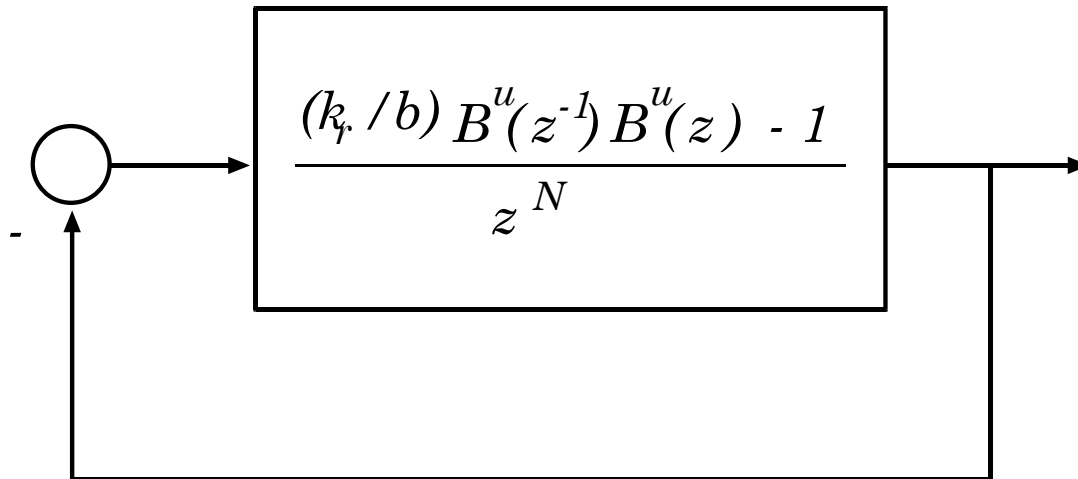


$$1 + \frac{\frac{k_r}{b} B^u(z) B^u(z^{-1}) - 1}{z^N} = 0$$

Closed-loop poles for non-minimum phase zeros

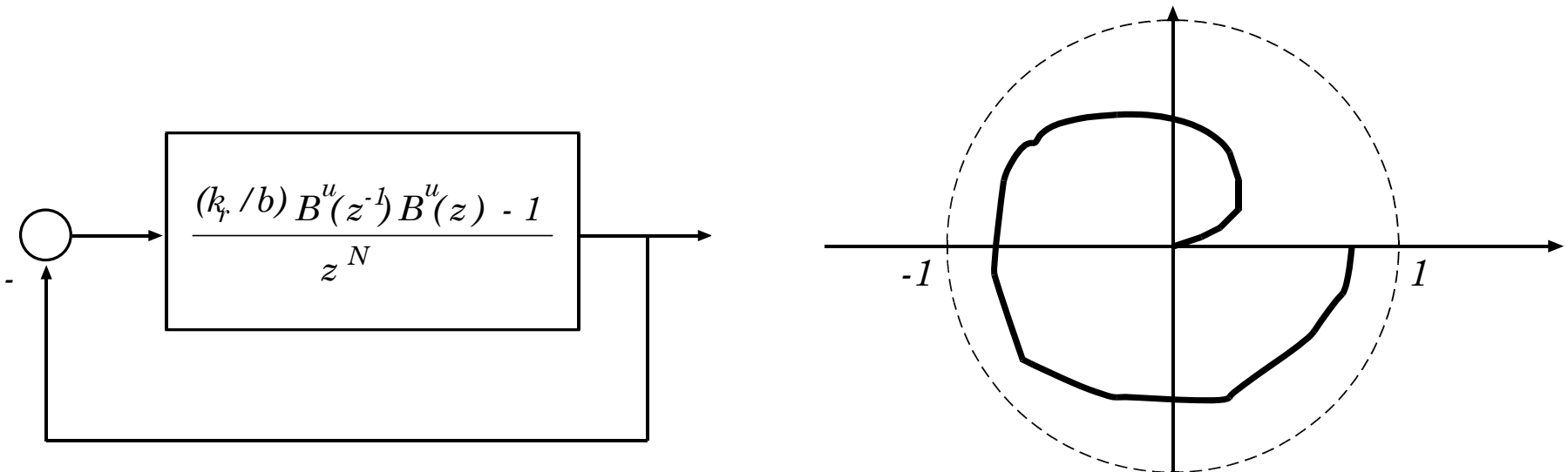
Therefore $\bar{A}_{cr}(z) = 0$ is equivalent to

$$1 + \frac{\frac{k_r}{b} B^u(z) B^u(z^{-1}) - 1}{z^N} = 0$$



Closed-loop poles for non-minimum phase zeros

By Nyquist's theorem, the closed-loop system is asymptotically stable if there are no encirclements around -1 .



This is guaranteed if the following hold for $\omega \in [0, \pi]$

$$\left| \frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| \leq 1$$

$$\frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \neq -1$$

Closed-loop poles for non-minimum phase zeros

Case 1: $B^u(e^{j\omega}) \neq 0$

We have $0 < \frac{|B^u(e^{j\omega})|^2}{b} = \frac{B^u(e^{j\omega}) B^u(e^{-j\omega})}{b} \leq 1$

$$2 > k_r > 0 \quad \Rightarrow \quad 0 < \frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) < 2$$

$$\Rightarrow \quad \left| \frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1 \right| < 1$$

$$|e^{j\omega N}| = 1 \quad \Rightarrow \quad \left| \frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| < 1$$

Closed-loop poles for non-minimum phase zeros

Case 2: $B^u(e^{j\omega}) = 0$

$$\text{We have } \left| \frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} \right| = \left| \frac{-1}{e^{j\omega N}} \right| = 1$$

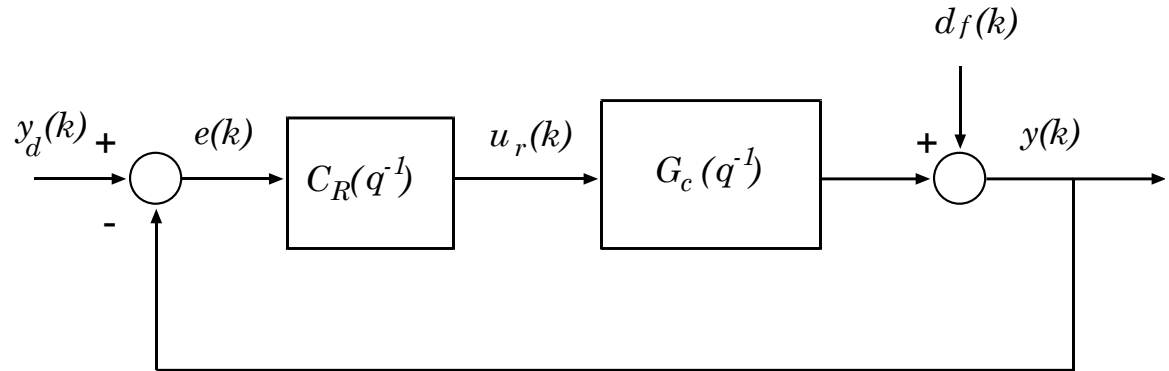
Since $B^u(q^{-1})$ and $1 - q^{-N}$ are co-prime, we have that

$$1 - e^{j\omega N} \neq 0 \quad \Rightarrow \quad e^{j\omega N} \neq 1$$

$$\Rightarrow \frac{\frac{k_r}{b} B^u(e^{j\omega}) B^u(e^{-j\omega}) - 1}{e^{j\omega N}} = \frac{-1}{e^{j\omega N}} \neq -1$$

\Rightarrow Closed-loop stability

Repetitive Compensator



Repetitive compensator:

$$C_R(q^{-1}) = \frac{k_r}{b} \left[\frac{q^{-N}}{1 - q^{-N}} \right] \left[q^d A'_c(q^{-1}) B^u(q) \right]$$

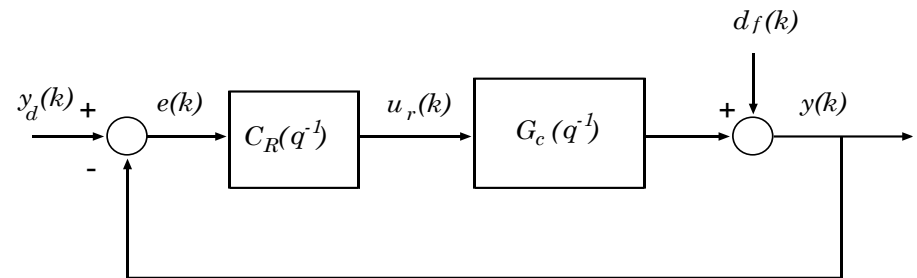
The controller has N open-loop poles on the unit circle

Repetitive control example

$$\begin{aligned} (B^u(q^{-1}) &= b_o) \\ (d &= 1) \end{aligned}$$

Assume that

$$N = 4$$



$$G_c(q^{-1}) = \frac{q^{-d} B^u(q^{-1})}{A'_c(q^{-1})} = \frac{b_o q^{-1}}{A'_c(q^{-1})}$$

Choose $b = b_o^2$

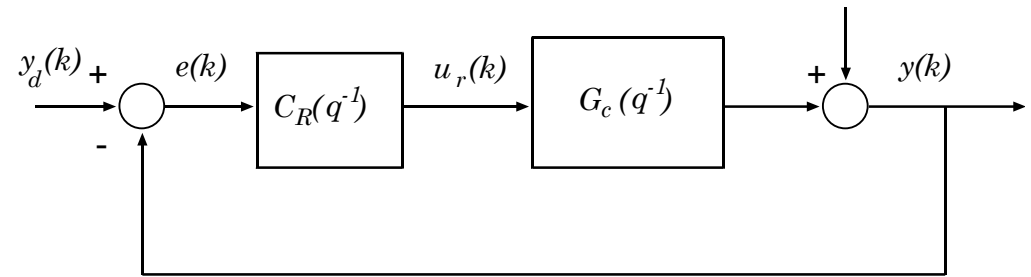
$$C_R(q^{-1}) = \frac{k_r}{b} q^{-(N-d)} \frac{A'_c(q^{-1}) B^u(q)}{1 - q^{-N}} = \frac{k_r}{b_o} q^{-3} \frac{A'_c(q^{-1})}{1 - q^{-4}}$$

$$\rightarrow G_c(q^{-1}) C_R(q^{-1}) = k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Repetitive control example

$$(B^u(q^{-1}) = b_o)$$

$$(d = 1)$$

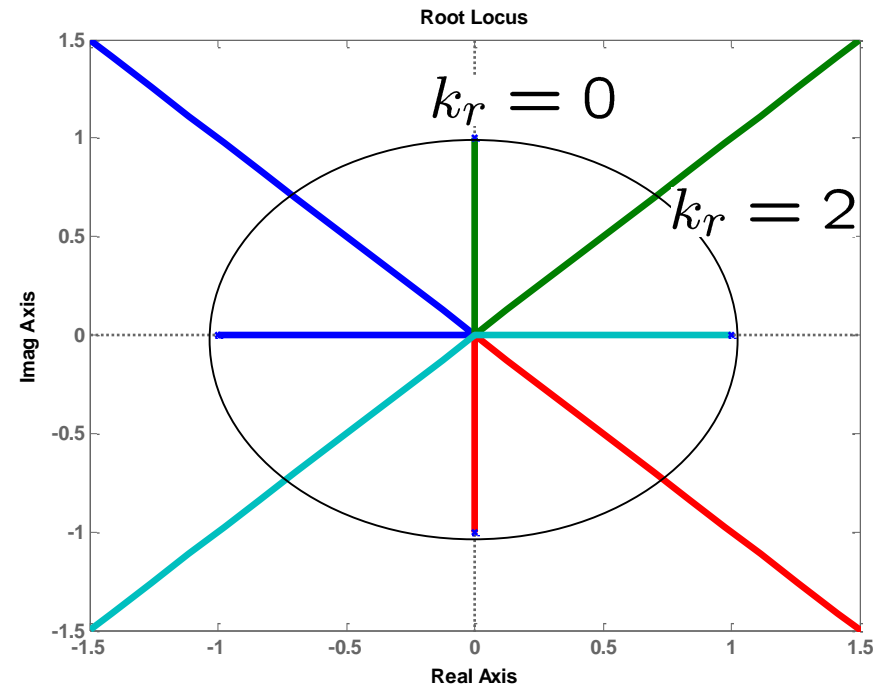


Open-loop TF

$$G_c(q^{-1})C_R(q^{-1}) = k_r \frac{q^{-4}}{1 - q^{-4}} = k_r \frac{1}{q^4 - 1}$$

Closed-loop poles:

$$1 + k_r \frac{1}{z^4 - 1} = 0$$

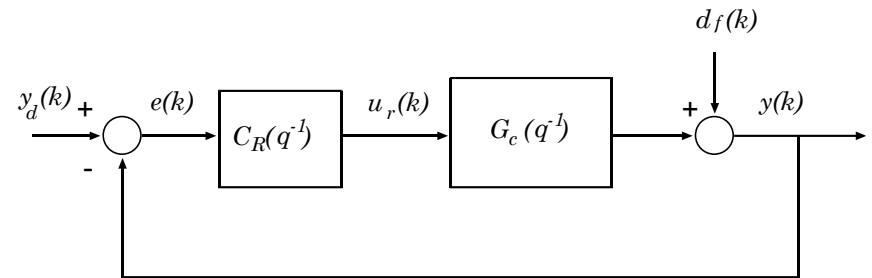


Repetitive control, inexact cancellation

Assume that

$$N = 4$$

Plant:



$$G_c(q^{-1}) = \frac{q^{-1}}{A'_c(q^{-1})} = \frac{q^{-1}}{\bar{A}'_c(q^{-1})} \frac{0.8 q^{-1}}{1 - 0.2 q^{-1}}$$

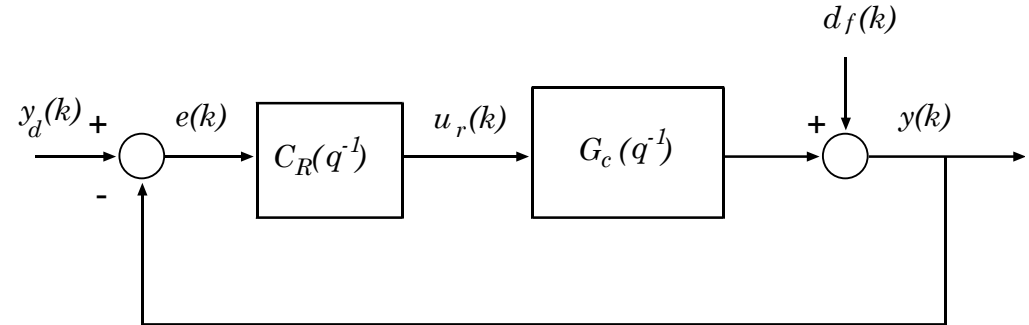
But, unmodeled dynamics are not cancelled →

$$C_R(q^{-1}) = \frac{k_r}{b_o} q^{-3} \frac{\bar{A}'_c(q^{-1})}{1 - q^{-4}}$$

therefore,

$$G_c(q^{-1})C_R(q^{-1}) = \frac{0.8 k_r}{(q - 0.2)(q^4 - 1)}$$

Repetitive control, inexact cancellation

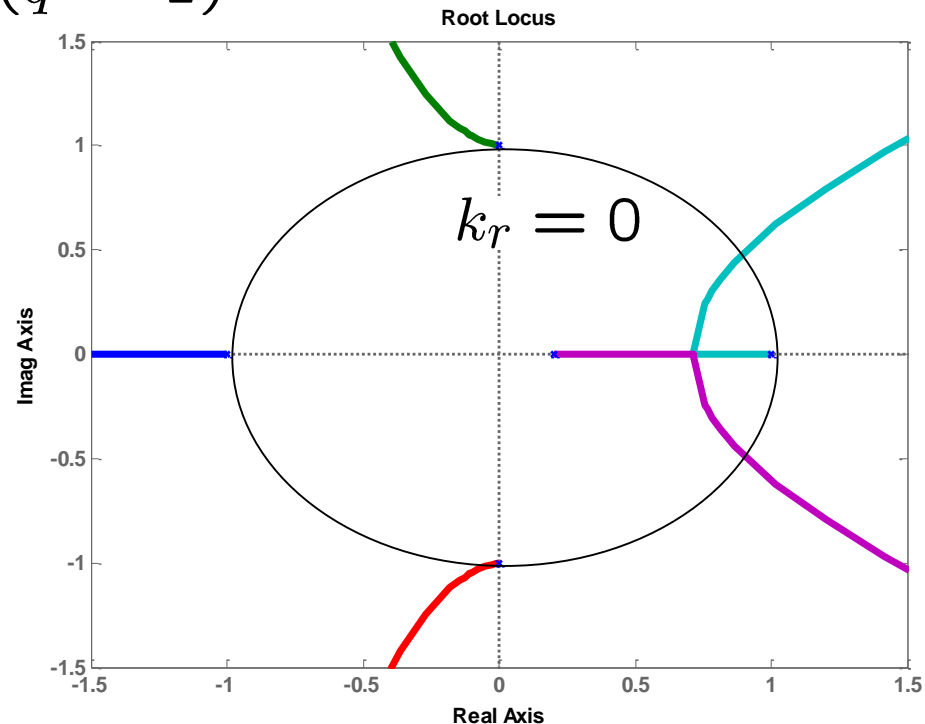


Open loop TF

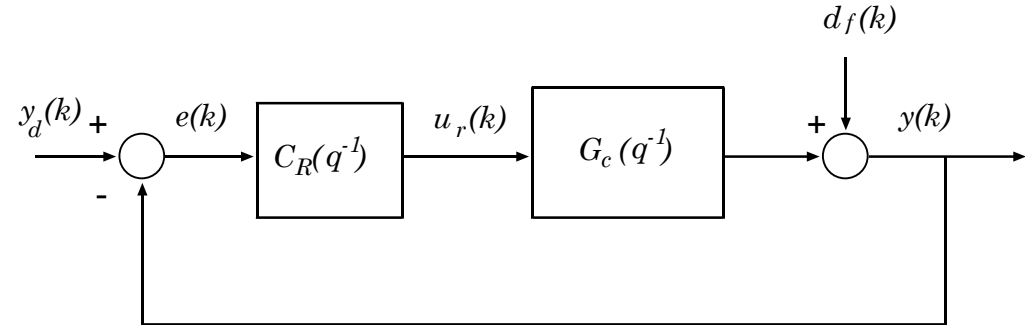
$$G_c(q^{-1})C_R(q^{-1}) = \frac{0.8 k_r}{(q - 0.2)(q^4 - 1)}$$

Closed-loop poles:

$$1 + k_r \frac{0.8}{(z - 0.2)(z^4 - 1)} = 0$$



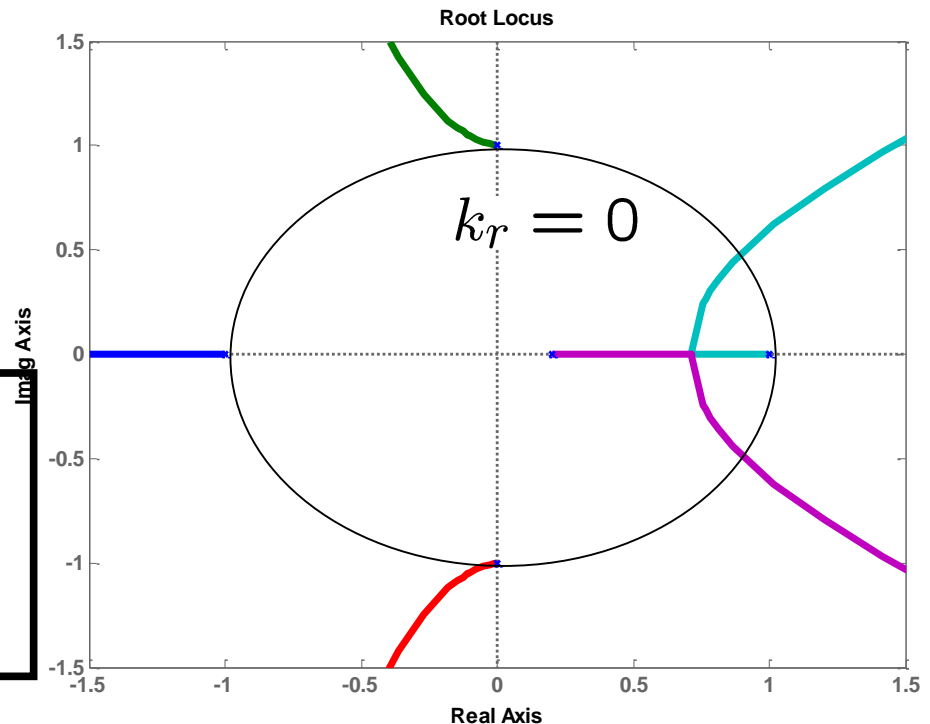
Repetitive control, inexact cancellation



Repetitive control is not robust to unmodeled dynamics

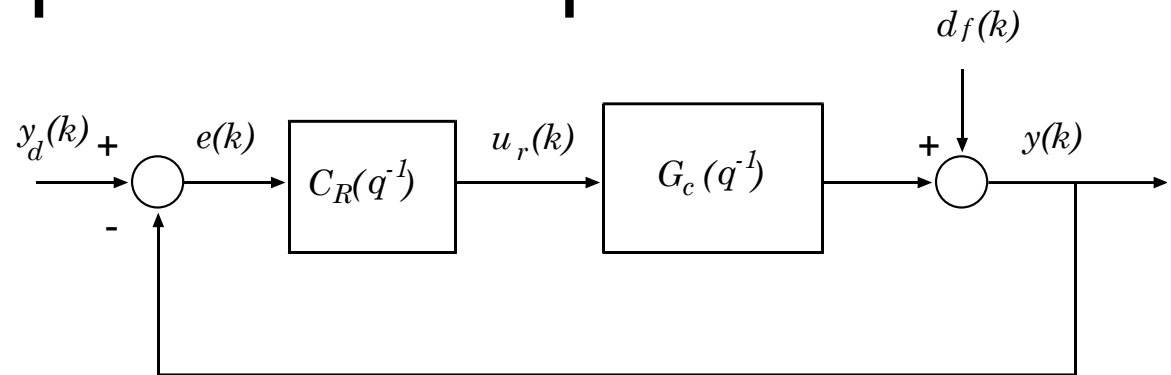
Closed-loop poles:

$$1 + k_r \frac{0.8}{(z - 0.2)(z^4 - 1)} = 0$$



Robust Repetitive Compensator

Add Q-filter



$$C_R(q^{-1}) = \frac{k_r}{b} q^{-(N-d)} \frac{A'_c(q^{-1}) B^u(q)}{1 - \underline{Q(q, q^{-1})} q^{-N}}$$

$Q(q, q^{-1})$ moving average filter with zero-phase shift characteristics

Controller's N open-loop poles are no longer on the unit circle

Robust Repetitive Compensator

$Q(q, q^{-1})$ moving average filter with zero-phase shift characteristics

$$Q(q, q^{-1}) = \frac{\gamma_p q^p + \cdots \gamma_1 q + \gamma_0 + \gamma_1 q^{-1} + \cdots \gamma_{p-1} q^{-(p-1)} + \gamma_p q^{-p}}{2\gamma_p + 2\gamma_{p-1} \cdots 2\gamma_1 + \gamma_0}$$

$$N > p \quad \gamma_0 > \gamma_1 > \cdots > \gamma_p > 0$$

$Q(q, q^{-1})$ has unit DC gain and gain decreases as frequency increases

Robust Repetitive Compensator

$$C_R(q^{-1}) = \frac{k_r}{b} q^{-(N-d)} \frac{A'_c(q^{-1})B^u(q)}{1 - \underline{Q(q, q^{-1})}q^{-N}}$$

Notice that the disturbance $d(k)$ is no longer completely annihilated, since

$$\left[1 - Q(q, q^{-1}) q^{-N} \right] d(k) \neq 0$$

However, with a proper choice of Q filter,

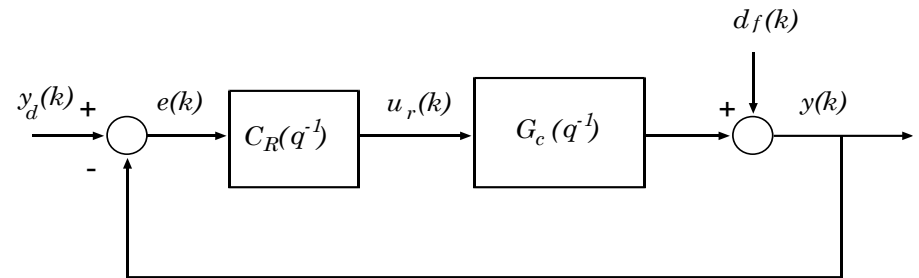
$$\left| \left[1 - Q(q, q^{-1}) q^{-N} \right] d(k) \right| \ll |d(k)|$$

Robust Rep. control, inexact cancellation

Assume that

$$N = 4$$

Plant:



$$G_c(q^{-1}) = \frac{q^{-1}}{A'_c(q^{-1})} = \frac{q^{-1}}{\bar{A}'_c(q^{-1})} \frac{0.8 q^{-1}}{1 - 0.2 q^{-1}}$$

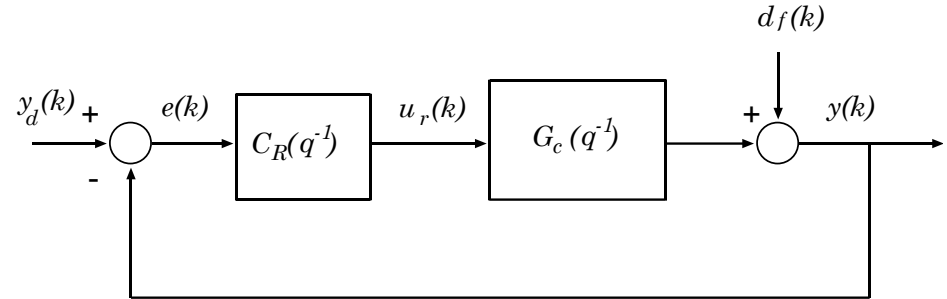
But, unmodeled dynamics are not cancelled →

$$C_R(q^{-1}) = \frac{k_r}{b_o} q^{-3} \frac{\bar{A}'_c(q^{-1})}{1 - Q(q, q^{-1}) q^{-4}}$$

where,

$$Q(q, q^{-1}) = \frac{q + 4 + q^{-1}}{6}$$

Robust Rep. control, inexact cancellation



Closed-loop poles:

$$1 + k_r \frac{2.4z}{(z - 0.2)(6z^5 - z^2 - 4z - 1)} = 0$$

Closed-loop system is asymptotically stable for a finite range of k_r

