ME 233 Advanced Control II

Lecture 14

Deterministic Input/Output Approach to SISO Discrete Time Systems

Pole Placement, Disturbance Rejection and Tracking Control

SISO ARMA models

- SISO State space model

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) \\
y(k) &= C\,x(k) + D\,u(k)
\end{aligned}
$$

Where all inputs and outputs are scalars:

- $u(k) \in \mathcal{R}$ control input
- $y(k) \in \mathcal{R}$ output
- $x(k) \in \mathcal{R}^n$ state

SISO transfer function

$$Y(z) = \left[C(zI - A)^{-1}B + D\right] U(z) \quad = \frac{B^*(z)}{A^*(z)} U(z)$$

$$A^*(z) \;=\; \mathrm{det}\{(zI - A)\} = z^n + a_1 z^{n-1} + \cdots + a_n$$

$$B^*(z) \;=\; C\,Adj\{(sI - A)\}B + D$$

$$= b_o\, z^m + b_1\, z^{n-1} + \cdots + b_m$$

$d = n - m \geq 0$ relative degree

SISO transfer function

$$Y(z) = \frac{\bar{B}(z)}{\bar{A}(z)} U(z)$$

U(z) control input
Y(z) output

$$\bar{A}(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

$$\bar{B}(z) = b_o\, z^m + b_1\, z^{n-1} + \cdots + b_m$$

$$\mathrm{d} = n - m \geq 0$$ relative degree

ARMA Models

Define:

- the ***back-step*** operator q^{-1} such that

$$y(k-1) = q^{-1}\, y(k)$$

- the polynomials

$$A(q^{-1}) = q^{-n}\,\bar{A}(q) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) = q^{-m}\,\bar{B}(q) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

- relative degree (pure time delay)

$$\mathrm{d} = n - m$$

Back-step operator

Relationship to Z-transform

$$\mathcal{Z}\{q^{-1}y(k)\} = \mathcal{Z}\{y(k-1)\} = z^{-1}Y(z)$$

Similarly,

$$\mathcal{Z}\{A(q^{-1})y(k)\} = A(z^{-1})Y(z)$$

SISO ARMA models

$$\underline{A(q^{-1})}\, y(k) = \underline{q^{-d}\, B(q^{-1})}\, u(k)$$

Auto-Regressive Moving Average

$$\begin{aligned} y(k) \;&=\; b_o u(k-d) + \cdots + b_m u(k-n) \\ &\quad -a_1 y(k-1) - \cdots - a_n y(k-n) \end{aligned}$$

SISO ARMA models with persistent disturbances

SISO ARMA model with disturbance

$$A(q^{-1})\,y(k) = q^{-\mathsf{d}}\,B(q^{-1})\,[\,u(k) + d(k)]$$

Where all inputs and outputs are scalars:

- $u(k)$ control input
- $d(k)$ persistent (deterministic) but unknown disturbance
- $y(k)$ output

Deterministic SISO ARMA models

$$A(q^{-1})\,y(k) = q^{-\mathsf{d}}\,B(q^{-1})\,[\,u(k) + d(k)]$$

Where polynomials:

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} \\ B(q^{-1}) &= b_o + b_1 q^{-1} + \cdots + b_m q^{-m} \end{aligned}$$

are co-prime and d is the ***known*** pure time delay

Polynomials in q^{-1}

- Monic polynomial

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

constant coefficient is 1

$$\bar{A}(q) = q^n A(q^{-1})$$

$$\bar{A}(q) = q^n + a_1 q^{n-1} + \cdots + a_n$$

leading coefficient is 1

monic

Polynomials in q^{-1}

- Co-prime polynomials have no common roots

The polynomials

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) = b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

are <u>co-prime</u> if and only if

$B(p) \neq 0$ whenever p satisfies $A(p) = 0$

Polynomials in q^{-1}

- Anti-Schur polynomials have all of their roots outside the unit circle

For example, if the polynomial

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

is anti-Schur, then $|q^{-1}| > 1$ whenever $A(q^{-1}) = 0$

Polynomials in q^{-1}

$$A(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

is anti-Schur if and only if

$$\lim_{k\to\infty} y(k) = 0$$

for all sequences $\{y(k)\} \in \mathcal{R}$ that satisfy

$$A(q^{-1})\, y(k) \;=\; 0$$

Factorization of the zero polynomial $B(q^{-1})$

Assume the m order zero polynomial $\bar{B}(q)$ has

- m_u zeros that **<u>we do not want to cancel</u>**.
- its remaining m_s zeros inside the unit circle; these are the zeros we **<u>will</u>** cancel

$$\bar{B}(q) = \bar{B}^s(q)\bar{B}^u(q)$$

$$\Rightarrow q^{-m}\bar{B}(q) = \left(q^{-m_s}\bar{B}^s(q)\right)\left(q^{-m_u}\bar{B}^u(q)\right)$$

$$B(q^{-1}) = B^s(q^{-1})\,B^u(q^{-1})$$

Factorization of the zero polynomial $B(q^{-1})$

Assume the m order zero polynomial $\bar{B}(q)$ has

- m_u zeros that **<u>we do not want to cancel</u>**.
- its remaining m_s zeros inside the unit circle; these are the zeros we **<u>will</u>** cancel

$$B(q^{-1}) = B^s(q^{-1})\,B^u(q^{-1})$$

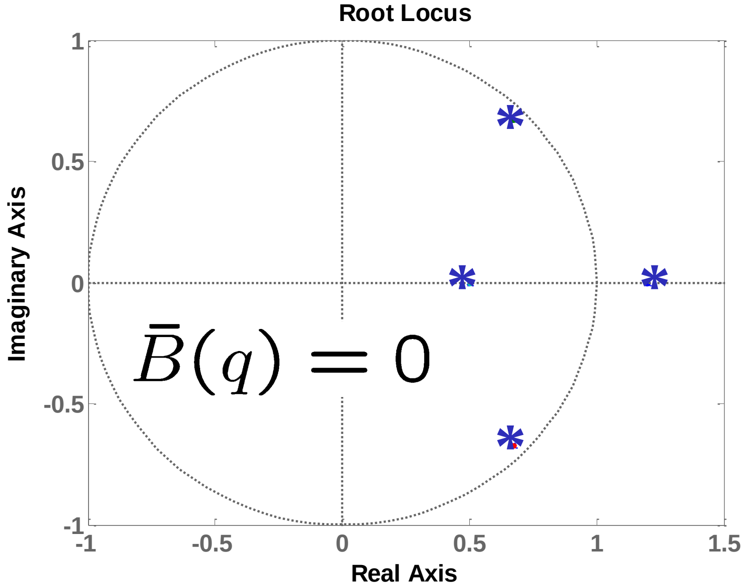
$B^s(q^{-1})$ is anti-Schur

$\bar{B}^u(q) = q^{m_u} B^u(q^{-1})$ has the zeros (in *q*) that we **<u>do not want to cancel</u>**

Example

$$\bar{B}(q) = 1.2(q-0.5)(q-1.2)(q-0.95e^{j\frac{\pi}{4}})(q-0.95e^{-j\frac{\pi}{4}})$$

$$B(q^{-1}) = q^{-4}\bar{B}(q)$$



Example

$$\bar{B}(q) = 1.2(q-0.5)(q-1.2)(q-0.95e^{j\frac{\pi}{4}})(q-0.95e^{-j\frac{\pi}{4}})$$

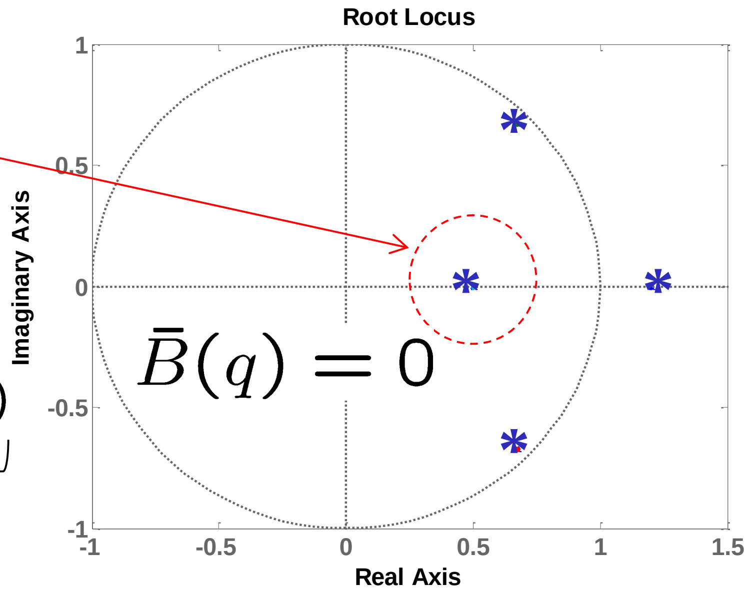
$$B(q^{-1}) = 1.2(1-0.5q^{-1})(1-1.2q^{-1}) \times(1-0.95e^{j\frac{\pi}{4}}q^{-1})(1-0.95e^{-j\frac{\pi}{4}}q^{-1})$$

$$= B^s(q^{-1})\, B^u(q^{-1})$$

Example

$$B^s(q^{-1}) = (1-0.5q^{-1})$$

$$B^u(q^{-1}) = 1.2(1-1.2q^{-1}) \underbrace{(1-0.95e^{j\frac{\pi}{4}}q^{-1})(1-0.95e^{-j\frac{\pi}{4}}q^{-1})}_{\text{we could have chosen to cancel these 2}}$$



Deterministic SISO ARMA models

The zero polynomial:

$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

Without loss of generality, we will assume that

$$B^s(q^{-1}) = 1 + \cdots + b^s_{m_s}\, q^{-m_s}$$

$$B^u(q^{-1}) = b_o + \cdots + b^u_{m_u}\, q^{-m_u}$$

Control Objectives

1. **<u>Pole Placement</u>**: The poles of the closed-loop system must be placed at specific locations in the complex plane.

- **Closed-loop polynomial:**

$$\boxed{A_c(q^{-1}) = B^s(q^{-1})\, A_c^{'}(q^{-1})}$$

Where:

chosen by the designer

- $B^s(q^{-1})$ cancelable plant zeros
- $A_c^{'}(q^{-1})$ anti-Schur polynomial of the form

$$A_c^{'}(q^{-1}) = 1 + a_{c1}^{'} q^{-1} + \cdots + a_{cn_c^{'}}^{'} q^{-n_c^{'}}$$

Control Objectives

2. **Tracking**: The output sequence $y(k)$ must follow a ***reference*** sequence $y_d(k)$ which is known

In general, $y_d(k)$ can be generated by a reference model of the form

$$A_m(q^{-1})y_d(k) = q^{-\mathrm{d}}\,B_m(q^{-1})\,u_d(k)$$

anti-Schur polynomial

The design of $A_m(q^{-1})$ and $B_m(q^{-1})$ is not a part of this control design technique and these polynomials do not enter into the analysis

Control Objectives

2. **<u>Tracking</u>**: The output sequence $y(k)$ must follow a ***reference*** sequence $y_d(k)$ which is known

- **Reference model:**

$$A_m(q^{-1})y_d(k) = q^{-\mathrm{d}}\,B_m(q^{-1})\,u_d(k)$$

Where:

- $y_d(k)$ **reference output sequence,** which is known in advance (i.e. $y_d(k+L)$ is available at instance k for some $L>d$).

- $A_m(q^{-1})$ anti-Schur polynomial

- $B_m(q^{-1})$ polynomial

chosen by the designer

Control Objectives

3. <u>Disturbance rejection</u>: The closed-loop system must reject a class of **<u>persistent</u>** disturbances $d(k)$

- **Disturbance model:**

$$\boxed{A_d(q^{-1})d(k) = 0}$$

Where

- $A_d(q^{-1})$ is a ***known*** annihilating polynomial <u>with zeros on the unit circle</u>

- $A_d(q^{-1}),\ B(q^{-1})$ are co-prime

Deterministic disturbance examples

a) Constant disturbance:

$$d(k) = d(k-1)$$

Then,

$$\boxed{A_d(q^{-1}) = 1 - q^{-1}}$$

b) Sinusoidal disturbance of ***<u>known</u>*** frequency:

$$d(k) = D\ \sin(\omega\, k + \phi)$$

Then,

$$\boxed{A_d(q^{-1}) = 1 - 2\cos(\omega)\, q^{-1} + q^{-2}}$$

Deterministic disturbance examples

c) Periodic disturbance of ***known*** period N

$$d(k) = d(k - N)$$

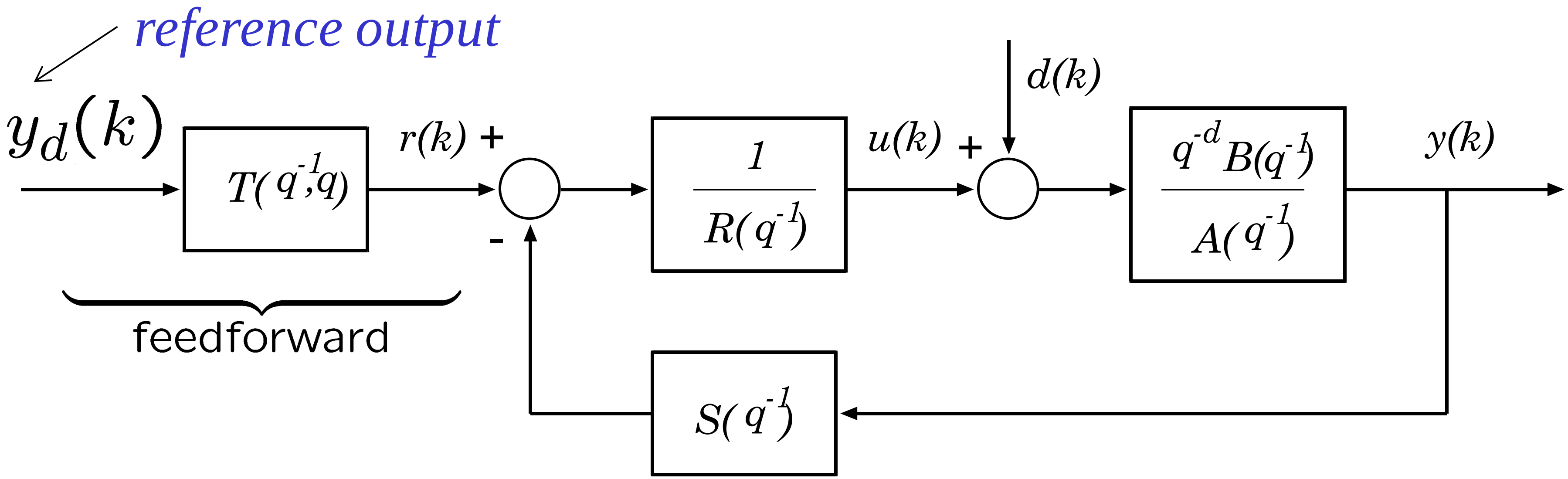
Then,

$$\boxed{A_d(q^{-1}) = 1 - q^{-N}}$$

In all of these three examples, the polynomial $A_d(q^{-1})$ has its roots on the unit circle.

Control Law

- Feedback and feedforward actions:

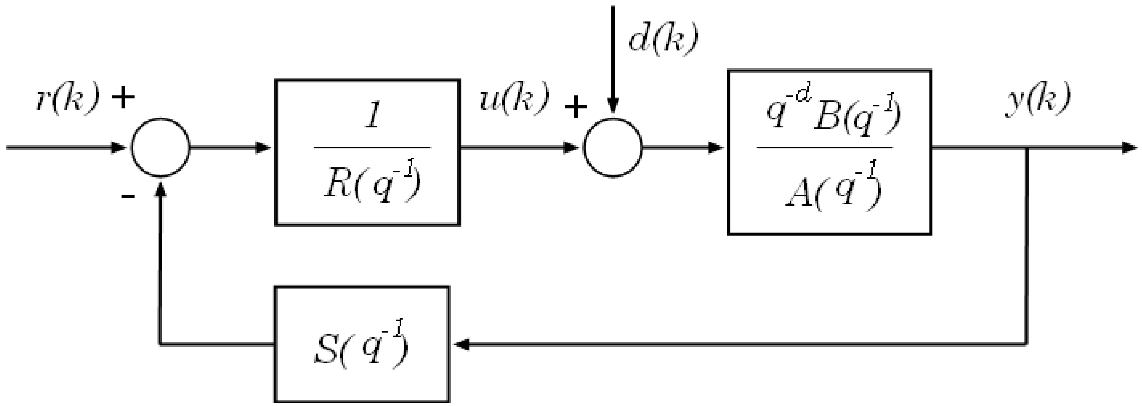


$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$r(k) = T(q^{-1}, q)\, y_d(k)$$

Feedforward action (a-causal)

Closed-Loop TF from *r(k)* to *y(k)*

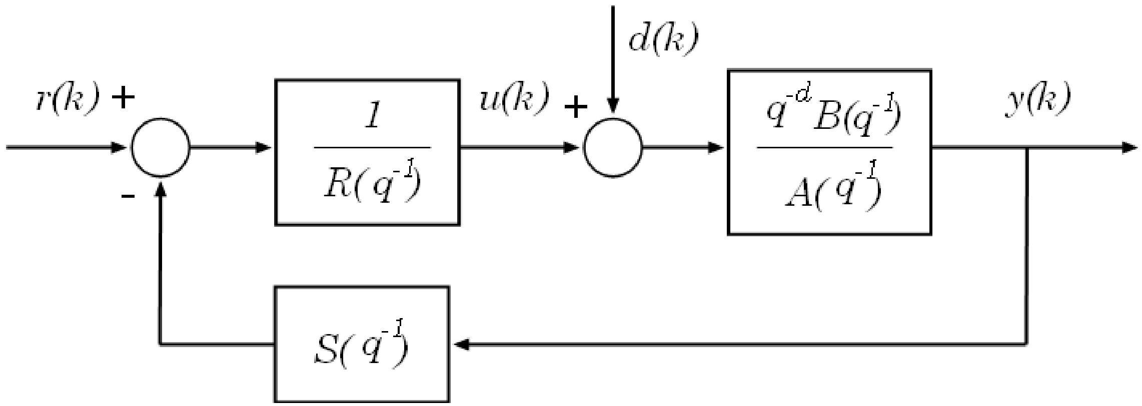


$$\mathrm{y(k)} = \frac{\frac{1}{R(q^{-1})}\frac{q^{-\mathsf{d}}B(q^{-1})}{A(q^{-1})}}{1+\frac{S(q^{-1})}{R(q^{-1})}\frac{q^{-\mathsf{d}}B(q^{-1})}{A(q^{-1})}}\; r(k) + \Big(\qquad\Big) d(k)$$

$$= \frac{q^{-\mathsf{d}}B(q^{-1})}{\underbrace{A(q^{-1})R(q^{-1}) + q^{-\mathsf{d}}B(q^{-1})S(q^{-1})}_{A_c(q^{-1})}}\; r(k) + \Big(\qquad\Big) d(k)$$

Closed-loop characteristic polynomial

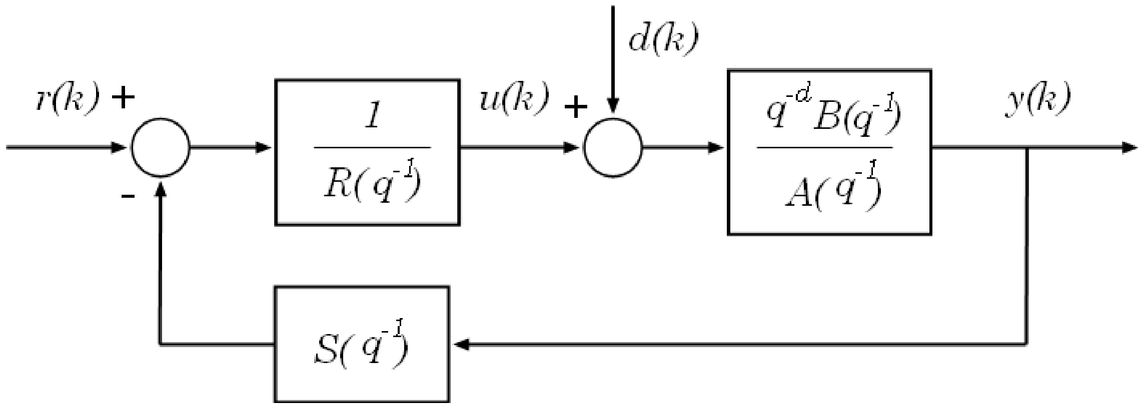
Closed-Loop TF from *r(k)* to *y(k)*



If we let $A_c(q^{-1})$ have the special structure $B^s(q^{-1})A'_c(q^{-1})$

$$y(k) = \frac{q^{-d}\cancel{B^s(q^{-1})}B^u(q^{-1})}{\cancel{B^s(q^{-1})}A'_c(q^{-1})}\, r(k) + \Big(\qquad \Big) d(k)$$

Closed-Loop TF from $r(k)$ to $y(k)$



$$y(k) = \underbrace{\frac{q^{-\mathrm{d}}B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1})}}_{A_c(q^{-1})}\; r(k) + \left(\qquad \right) d(k)$$

Closed-loop characteristic polynomial

Given $A_c(q^{-1})$, we would like to find polynomials $R(q^{-1})$ and $S(q^{-1})$ so that

$$A(q^{-1})R(q^{-1}) + q^{-\mathrm{d}}B(q^{-1})S(q^{-1}) = A_c(q^{-1})$$

The Diophantine (Bezout) equation

- Given the ***co-prime*** polynomials

$$\mathcal{A}(q^{-1}) \;=\; 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$\mathcal{B}(q^{-1}) \;=\; b_o + b_1 q^{-1} + \cdots + b_m q^{-m}$$

  - $\mathcal{A}(q^{-1})$ is order n and has constant term 1
  - $\mathcal{B}(q^{-1})$ is order m

- and a polynomial of order n_c with constant term 1

$$\mathcal{C}(q^{-1}) \;=\; 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}$$

The Diophantine (Bezout) equation

We wish to find polynomials

same order as $\mathcal{B}(q^{-1})$

$$\mathcal{R}(q^{-1}) = 1 + r_1 q^{-1} + \cdots + r_m q^{-m}$$

$$\mathcal{S}(q^{-1}) = s_o + \cdots + s_{n_s} q^{-n_s}$$

that satisfy the Diophantine equation:

$$\boxed{\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})}$$

given

The Diophantine (Bezout) equation

Expanding in terms of q^{-1} coefficients:

$$\underbrace{\mathcal{C}(q^{-1})}_{\text{order } n_c} = \underbrace{\mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1})}_{\text{order } n+m} + \underbrace{q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})}_{\text{order } 1+m+n_s}$$

$$\boxed{n_s = \max\{n-1, n_c - m - 1\}}$$

The Diophantine (Bezout) equation

Expanding in terms of q^{-1} coefficients:

$$\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$$

We obtain:

$$\begin{array}{c} q^{-1} \\ q^{-2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ q^{-(n_s+m+1)} \end{array} \qquad D \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \\ s_o \\ \vdots \\ s_{n_s} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{nc} \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$D \in \mathcal{R}^{(n_s+1+m)\times(n_s+1+m)}$ given on next slide

The Diophantine (Bezout) equation

Where the matrix $D \in \mathcal{R}^{(n_s+1+m)\times(n_s+1+m)}$

is given by:

$$D = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & b_0 & 0 & 0 & \cdots & 0 \\
a_1 & 1 & 0 & \cdots & 0 & b_1 & b_0 & 0 & \cdots & 0 \\
a_2 & a_1 & 1 & \cdots & 0 & b_2 & b_1 & b_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 & b_m & b_{m-1} & b_{m-2} & \cdots & b_o \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & 0 & b_m & b_{m-1} & \cdots & b_1 \\
0 & a_n & a_{n-1} & \cdots & a_2 & 0 & 0 & b_m & \cdots & b_2 \\
0 & 0 & a_n & \cdots & a_3 & 0 & 0 & 0 & \cdots & b_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & a_n & 0 & 0 & 0 & \cdots & b_m
\end{bmatrix}$$

m $\qquad\qquad\qquad$ $n_s + 1$

$$
D = \left[\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots \\ a_2 & a_1 & \ddots & 0 \\ \vdots & a_2 & \ddots & 1 \\ a_{n-1} & \vdots & \ddots & a_1 \\ a_n & a_{n-1} & \ddots & a_2 \\ 0 & a_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & a_n \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}}_{m \text{ columns}} \quad \underbrace{\begin{bmatrix} b_0 & 0 & \cdots & 0 & 0 \\ b_1 & b_0 & \ddots & \vdots & \vdots \\ \vdots & b_1 & \ddots & 0 & 0 \\ b_{m-1} & \vdots & \ddots & b_0 & 0 \\ b_m & b_{m-1} & \ddots & b_1 & b_0 \\ 0 & b_m & \ddots & \vdots & b_1 \\ 0 & 0 & \ddots & b_{m-1} & \vdots \\ \vdots & \vdots & \ddots & b_m & b_{m-1} \\ 0 & 0 & \ddots & 0 & b_m \end{bmatrix}}_{n_s + 1 \text{ columns}} \right]
$$

$$D = \left[\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots \\ a_2 & a_1 & \ddots & 0 \\ \vdots & a_2 & \ddots & 1 \\ a_{n-1} & \vdots & \ddots & a_1 \\ a_n & a_{n-1} & \ddots & a_2 \\ 0 & a_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & a_n \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad \begin{bmatrix} b_0 & 0 & \cdots & 0 & 0 \\ b_1 & b_0 & \ddots & \vdots & \vdots \\ \vdots & b_1 & \ddots & 0 & 0 \\ b_{m-1} & \vdots & \ddots & b_0 & 0 \\ b_m & b_{m-1} & \ddots & b_1 & b_0 \\ 0 & b_m & \ddots & \vdots & b_1 \\ 0 & 0 & \ddots & b_{m-1} & \vdots \\ \vdots & \vdots & \ddots & b_m & b_{m-1} \\ 0 & 0 & \ddots & 0 & b_m \end{bmatrix} \right]$$

If $n_s = n - 1$, then these rows of zeros will not be here

The Diophantine (Bezout) equation

Theorem: D is nonsingular iff the polynomials $\mathcal{A}(q^{-1})$ and $q^{-1}\mathcal{B}(q^{-1})$ are co-prime.

The solution to the Diophantine equation is:

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \\ s_o \\ \vdots \\ s_{n_s} \end{bmatrix} = D^{-1} \left\{ \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{nc-1} \\ c_{nc} \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} .$$

Example: $\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$

Let

$$\mathcal{C}(q^{-1}) = (1 - 0.5q^{-1})(1 - 0.8q^{-1}) \qquad \text{order } n_c = 2$$
$$= (1 - 1.3q^{-1} + 0.4q^{-2})$$

$$\mathcal{A}(q^{-1}) = (1 - q^{-1})(1 - 1.2q^{-1}) \qquad \text{order } n = 2$$
$$= (1 - 2.2q^{-1} + 1.2q^{-2})$$

$$\mathcal{B}(q^{-1}) = (2q^{-1} + 2.4q^{-2}) \qquad \text{order } m = 2$$

Solve for
$$\left\{\begin{array}{ll} \mathcal{R}(q^{-1}) = 1 + r_1 q^{-1} + r_2 q^{-2} & \text{order } m = 2 \\ \mathcal{S}(q^{-1}) & \text{order } n_s \end{array}\right.$$

Example: $\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$

$$n_s = \max\{n-1, n_c - m - 1\} \; = \max\{2-1, 2-2-1\} = 1$$

$$\underbrace{1 - 1.3q^{-1} + 0.4q^{-2}}_{\mathcal{C}(q^{-1})} = (\underbrace{1 - 2.2q^{-1} + 1.2q^{-2}}_{\mathcal{A}(q^{-1})})\,(\underbrace{1 + r_1 q^{-1} + r_2 q^{-2}}_{\mathcal{R}(q^{-1})})$$

$$+q^{-1}(\underbrace{2q^{-1} + 2.4q^{-2}}_{\mathcal{B}(q^{-1})})\;(\underbrace{s_o + s_1 q^{-1}}_{\mathcal{S}(q^{-1})})$$

4 equations and 4 unknowns

Example: $\mathcal{C}(q^{-1}) = \mathcal{A}(q^{-1})\,\mathcal{R}(q^{-1}) + q^{-1}\,\mathcal{B}(q^{-1})\mathcal{S}(q^{-1})$

Equating coefficients of powers of q^{-1}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2.2 & 1 & 2 & 0 \\ 1.2 & -2.2 & 2.4 & 2 \\ 0 & 1.2 & 0 & 2.4 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} -1.3 \\ 0.4 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2.2 \\ 1.2 \\ 0 \\ 0 \end{bmatrix}$$

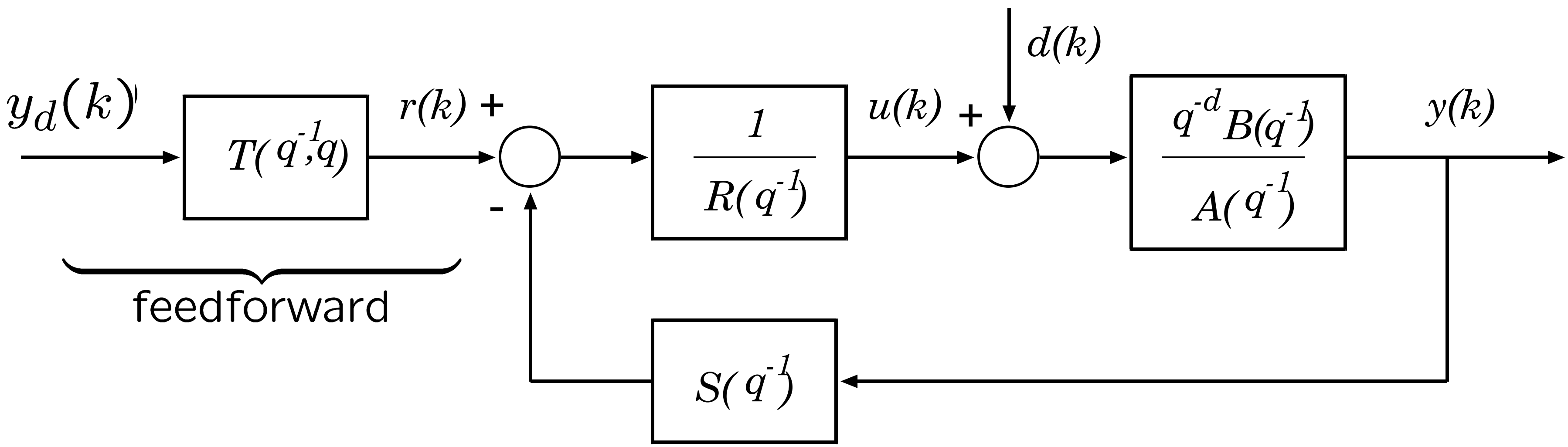
Solution:

$$\mathcal{R}(q^{-1}) = 1 + 0.9q^{-1} + 0.57q^{-2}$$

$$\mathcal{S}(q^{-1}) = 0.31 - 0.28q^{-1}$$

Return to the Control Problem…

- Feedback and feedforward actions:



$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

$$r(k) = T(q^{-1}, q)\, y_d(k)$$ *Feedforward (a-causal)*

Feedback Controller

Diophantine equation: Obtain polynomials $R(q^{-1}),\ S(q^{-1})$ that satisfy:

$$A_c(q^{-1}) = A(q^{-1})\,\underline{R(q^{-1})} + q^{-d}B(q^{-1})\,\underline{S(q^{-1})}$$

Closed-loop poles

Plant poles

plant zeros

$$A_c(q^{-1}) = \underline{B^s(q^{-1})}\,A_c^{'}(q^{-1})$$

$$R(q^{-1}) = R^{'}(q^{-1})\,\underline{A_d(q^{-1})}\,\underline{B^s(q^{-1})}$$

We will factor out the $B^s(q^{-1})$ *polynomial next*

Controller Diophantine equation

Factor out $B^s(q^{-1})$ polynomial

$$A_c(q^{-1}) = B^s(q^{-1})\,A_c^{'}(q^{-1})$$

$$R(q^{-1}) = R^{'}(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1})$$

$$A_c(q^{-1}) = A(q^{-1})\,R(q^{-1}) + q^{-\mathsf{d}}B(q^{-1})\,S(q^{-1})$$

$$\cancel{B^s(q^{-1})}\,A_c^{'}(q^{-1}) = \cancel{B_s(q^{-1})}A(q^{-1})A_d(q^{-1})R^{'}(q^{-1})$$
$$+q^{-\mathsf{d}}\cancel{B^s(q^{-1})}B^u(q^{-1})\,S(q^{-1})$$

Therefore, we want to find $R'(q^{-1})$ and $S(q^{-1})$ such that

$$A_c^{'}(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R^{'}(q^{-1}) + q^{-\mathsf{d}}B^u(q^{-1})\,S(q^{-1})$$

Feedback Controller

Diophantine equation: Obtain polynomials $\underline{R'(q^{-1})}$, $\underline{S(q^{-1})}$ which satisfy:

$$\boxed{A_c'(q^{-1}) = A_d(q^{-1})\, A(q^{-1})\, \underline{R'(q^{-1})} + q^{-\mathrm{d}} B^u(q^{-1})\, \underline{S(q^{-1})}}$$

Closed-loop poles without cancelled zeros

Plant poles

Unstable plant zeros

Disturbance annihilating polynomial

$$\boxed{\begin{aligned} R(q^{-1}) &= R'(q^{-1})\, A_d(q^{-1})\, B^s(q^{-1}) \\ A_c(q^{-1}) &= B^s(q^{-1})\, A_c'(q^{-1}) \end{aligned}}$$

Use previous solution of the Diophantine equation

$$A_c'(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R'(q^{-1}) + q^{-\mathsf{d}}B^u(q^{-1})\,S(q^{-1})$$

$$\underbrace{A_c'(q^{-1})}_{\mathcal{C}(q^{-1})} = \underbrace{\Big(A_d(q^{-1})\,A(q^{-1})\Big)}_{\mathcal{A}(q^{-1})}\ \underbrace{R'(q^{-1})}_{\mathcal{R}(q^{-1})}$$

$$+q^{-1}\underbrace{\Big(q^{-(\mathsf{d}-1)}B^u(q^{-1})\Big)}_{\mathcal{B}(q^{-1})}\underbrace{S(q^{-1})}_{\mathcal{S}(q^{-1})}$$

Diophantine equation

$$A_c'(q^{-1}) = A_d(q^{-1})\,A(q^{-1})\,R'(q^{-1}) + q^{-\mathsf{d}}B^u(q^{-1})\,S(q^{-1})$$

Solution:

$$R'(q^{-1}) \;=\; 1 + r_1'q^{-1} + \cdots + r_{n_r'}\,q^{-n_r'}$$

These are the controller parameters!

$$\begin{cases} S(q^{-1}) \;=\; s_o + s_1q^{-1} + \cdots + s_{n_s}q^{-n_s} \\ R(q^{-1}) = R'(q^{-1})\,A_d(q^{-1})\,B^s(q^{-1}) \end{cases}$$

$$\boxed{\begin{aligned} n_r' &= \mathsf{d} + m_u - 1 \\ n_s &= \max\{\,n + n_d - 1\,,\; n_c' - \mathsf{d} - m_u\,\} \\ n_r &= n_r' + n_d + m_s \end{aligned}}$$

Feedback Controller

$$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

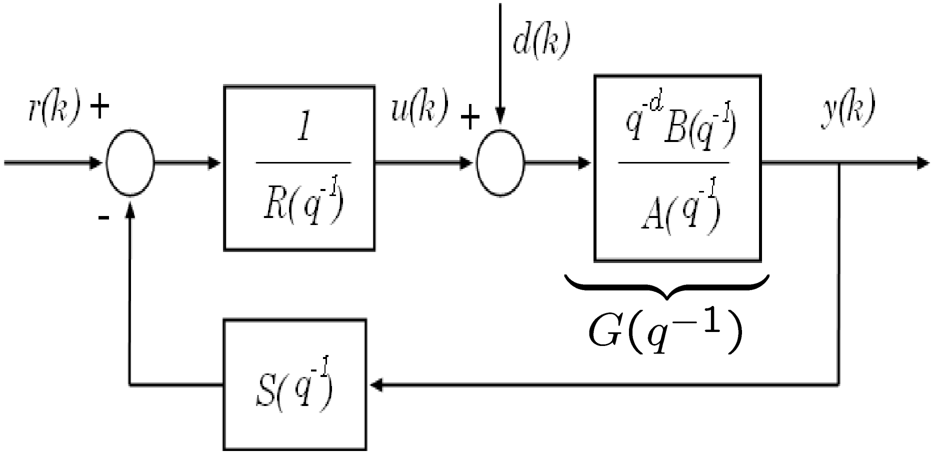
where

$$\boxed{\begin{aligned} n_r^{'} &= \mathsf{d} + m_u - 1 \\ n_s &= \max\{\, n + n_d - 1 \,,\, n_c^{'} - \mathsf{d} - m_u \,\} \\ n_r &= n_r^{'} + n_d + m_s \end{aligned}}$$

If the degree of the disturbance annihilator polynomial, n_d is large (e.g. N is large), then n_r and n_s are also large

Then, the solution of the Diophantine equation may be ill conditioned.

Example



$$G(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1}))}$$

Plant:

$$G(q^{-1}) = \frac{q^{-2}(2+2.4q^{-1})}{(1-1.2q^{-1})}$$

Zeros: $B(q^{-1}) = (2+2.4q^{-1})$ ⇒ $\begin{cases} B^s(q^{-1}) = 1 \\ B^u(q^{-1}) = (2+2.4q^{-1}) \end{cases}$

Disturbance: $d(k) = d(k-1)$ ⇒ $A_d(q^{-1}) = 1 - q^{-1}$

Select closed-loop poles:

$$A_c'(q^{-1}) = (1-0.5q^{-1})(1-0.8q^{-1})$$

$$= (1-1.3q^{-1}+0.4q^{-2})$$

Diophantine equation

$$A_c'(q^{-1}) = A_d(q^{-1})\, A(q^{-1})\, R'(q^{-1}) + q^{-\mathrm{d}} B^u(q^{-1})\, S(q^{-1})$$

$$(1 - 1.3q^{-1} 0.4q^{-2}) = \underbrace{(1 - 2.2q^{-1} + 1.2q^{-2})}_{A(q^{-1}) A_d(q^{-1})} \underbrace{(1 + r_1' q^{-1} + r_2' q^{-2})}_{R'(q^{-1})}$$

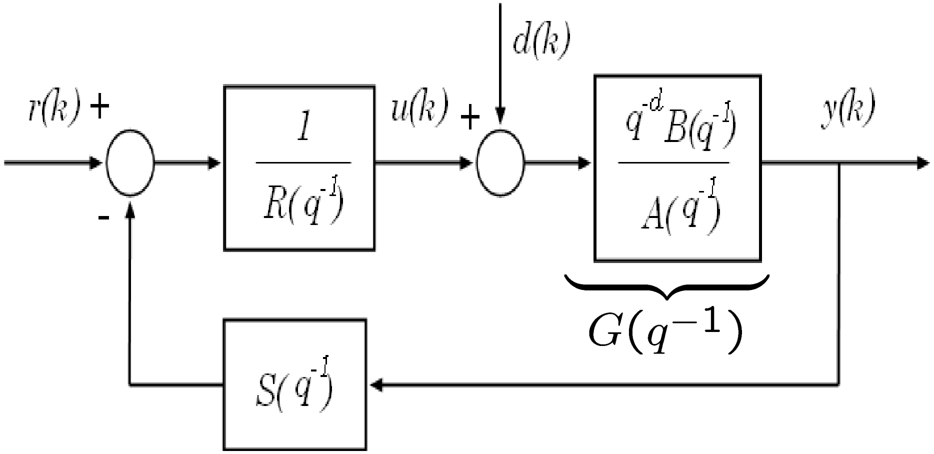
$$+ q^{-2} \underbrace{(2 + 2.4q^{-1})}_{B^u(q^{-1})} \quad \underbrace{(s_o + s_1 q^{-1})}_{S(q^{-1})}$$

Solution:

$$\left\{ \begin{aligned} R'(q^{-1}) &= 1 + 0.9q^{-1} + 0.57q^{-2} \\ S(q^{-1}) &= 0.31 - 0.28q^{-1} \end{aligned} \right.$$

$$\begin{aligned} R(q^{-1}) &= B^s(q^{-1}) A_d(q^{-1}) R'(q^{-1}) \\ &= (1 - q^{-1})(1 + 0.9q^{-1} + 0.57q^{-2}) \end{aligned}$$

Example



$$G(q^{-1}) = \frac{q^{-2}(2+2.4q^{-1})}{(1-1.2q^{-1})}$$

$$d(k) = d(k-1)$$

Control: $$u(k) = \frac{1}{R(q^{-1})}\left[r(k) - S(q^{-1})y(k)\right]$$

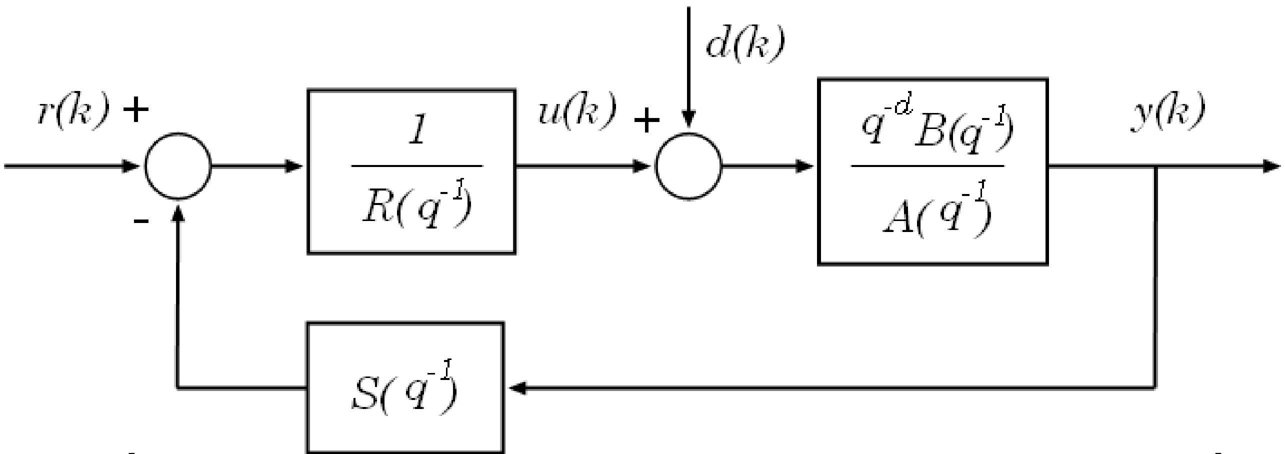
$$R(q^{-1}) = 1 - 0.1q^{-1} - 0.33q^{-2} - 0.57q^{-3}$$

$$S(q^{-1}) = 0.31 - 0.28q^{-1}$$

$$r(k) = r(k-1) = 1$$

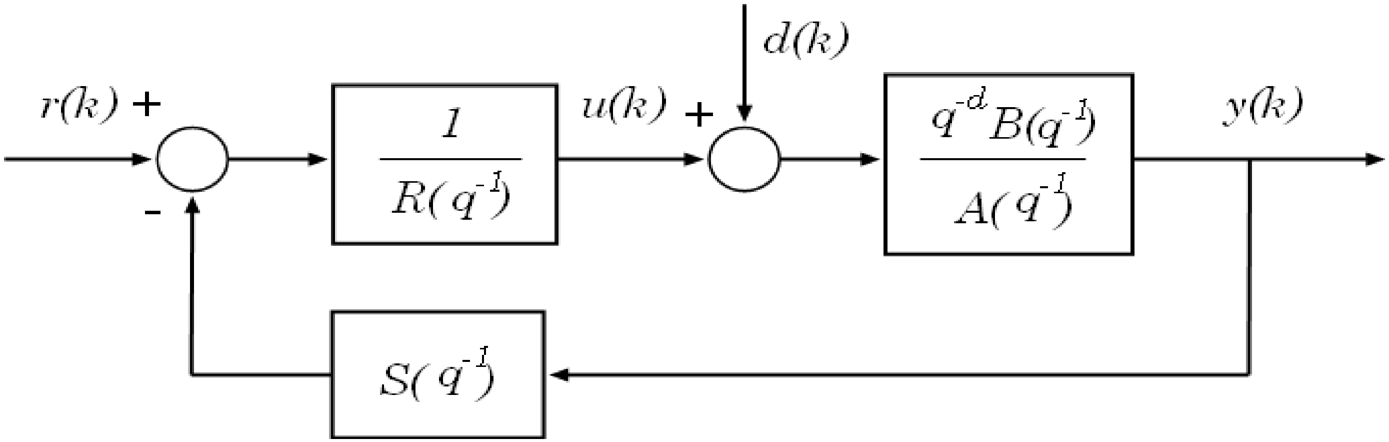
Feedback Control Law

Feedback control action:



$$y(k) = \frac{q^{-\mathsf{d}} B^u(q^{-1})}{A_c'(q^{-1})} r(k)$$

Proof – block diagram algebra



The closed-loop dynamics is from *r(k)* and *d(k)* to *y(k)*

$$y(k) = \frac{q^{-\mathsf{d}}B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathsf{d}}B(q^{-1})S(q^{-1})}\, r(k)$$

$$+ \frac{q^{-\mathsf{d}}B(q^{-1})\, R(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathsf{d}}B(q^{-1})S(q^{-1})}\, d(k)$$

Proof – block diagram algebra

The closed-loop dynamics from $d(k)$ to $y(k)$ ($r(k) = 0$)

$$y(k) = \frac{q^{-\mathsf{d}} B(q^{-1})\, R(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathsf{d}} B(q^{-1}) S(q^{-1})}\, d(k)$$

Substitute:

$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

$$R(q^{-1}) = R'(q^{-1})\, A_d(q^{-1})\, B^s(q^{-1})$$

$$y(k) = \cancelto{1}{\frac{B^s(q^{-1})}{B^s(q^{-1})}}\; \frac{q^{-\mathsf{d}} B(q^{-1}) R'(q^{-1}) A_d(q^{-1})}{\underbrace{\left[A(q^{-1}) A_d(q^{-1}) R'(q^{-1}) + q^{-\mathsf{d}} B^u(q^{-1}) S(q^{-1})\right]}_{A'_c(q^{-1})}}\, d(k)$$

pole-zero cancellation

Diophantine equation

Proof – block diagram algebra

The closed-loop dynamics from $d(k)$ to $y(k)$ ($r(k) = 0$)

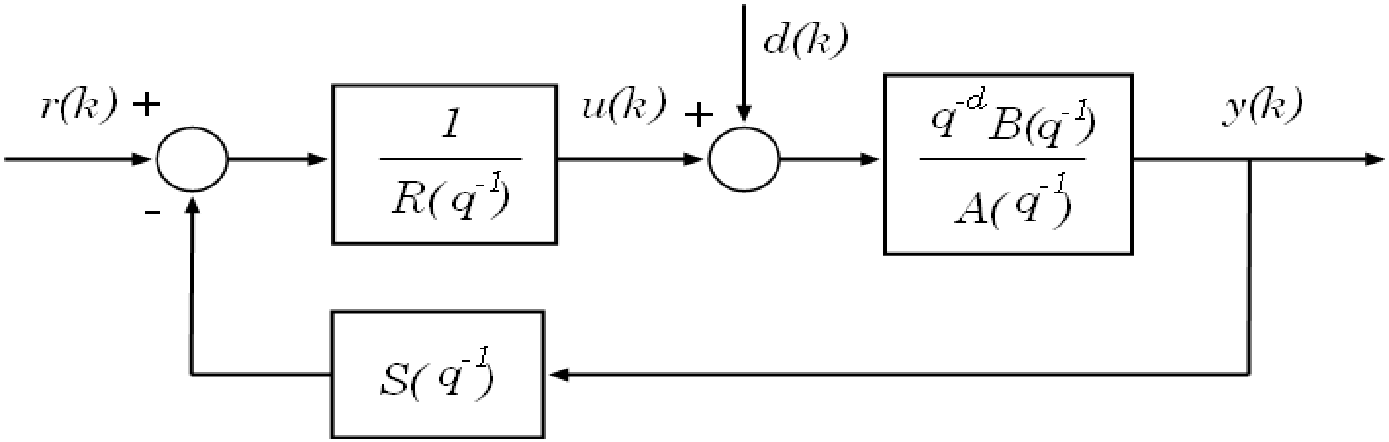
$$y(k) = \left[\frac{q^{-\mathsf{d}}B(q^{-1})R'(q^{-1})}{A_c'(q^{-1})}\right]\underbrace{A_d(q^{-1})d(k)}_{0}$$

$$y(k) = \cancelto{1}{\frac{B^s(q^{-1})}{B^s(q^{-1})}}\,\frac{q^{-\mathsf{d}}B(q^{-1})R'(q^{-1})A_d(q^{-1})}{\Big[\underbrace{A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathsf{d}}B^u(q^{-1})S(q^{-1})}_{A_c'(q^{-1})}\Big]}\,d(k)$$

pole-zero cancellation

Diophantine equation

Proof – block diagram algebra



The closed-loop dynamics from $r(k)$ and $d(k)$ to $y(k)$

$$y(k) = \frac{q^{-\mathsf{d}}B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathsf{d}}B(q^{-1})S(q^{-1})}\,r(k)$$

$$+ \underbrace{\frac{q^{-\mathsf{d}}B(q^{-1})\,R(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathsf{d}}B(q^{-1})S(q^{-1})}\,d(k)}_{\to 0}$$

Proof – block diagram algebra

$$y(k) = \frac{q^{-\mathsf{d}} B(q^{-1})}{A(q^{-1})R(q^{-1}) + q^{-\mathsf{d}} B(q^{-1}) S(q^{-1})}\, r(k)$$

Substitute:

$$B(q^{-1}) = B^s(q^{-1})\, B^u(q^{-1})$$

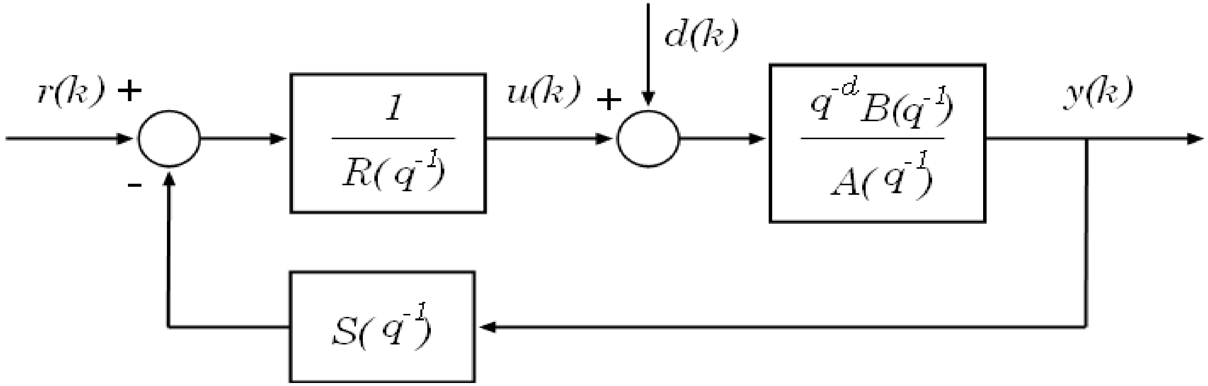
$$R(q^{-1}) = R^{'}(q^{-1})\, A_d(q^{-1})\, B^s(q^{-1})$$

$$y(k) = \cancelto{1}{\frac{B^s(q^{-1})}{B^s(q^{-1})}}\; \frac{q^{-\mathsf{d}} B^u(q^{-1})}{\underbrace{\left[A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathsf{d}} B^u(q^{-1}) S(q^{-1})\right]}_{A_c'(q^{-1})}}\, r(k)$$

pole-zero cancellation

Diophantine equation

Proof – block diagram algebra



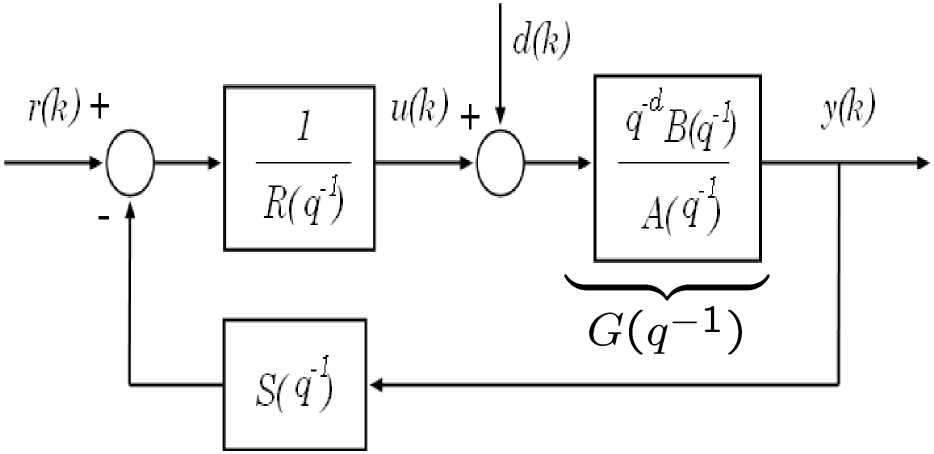
$$\boxed{y(k) = \frac{q^{-\mathrm{d}}B^u(q^{-1})}{A_c'(q^{-1})}\,r(k)}$$

$$y(k) = \cancelto{1}{\frac{B^s(q^{-1})}{B^s(q^{-1})}}\;\frac{q^{-\mathrm{d}}B^u(q^{-1})}{\underbrace{\left[A(q^{-1})A_d(q^{-1})R'(q^{-1}) + q^{-\mathrm{d}}B^u(q^{-1})S(q^{-1})\right]}_{A_c'(q^{-1})}}\,r(k)$$

pole-zero cancellation

Diophantine equation

Example



$$G(q^{-1}) = \frac{q^{-2}(2 + 2.4q^{-1})}{(1 - 1.2q^{-1})}$$

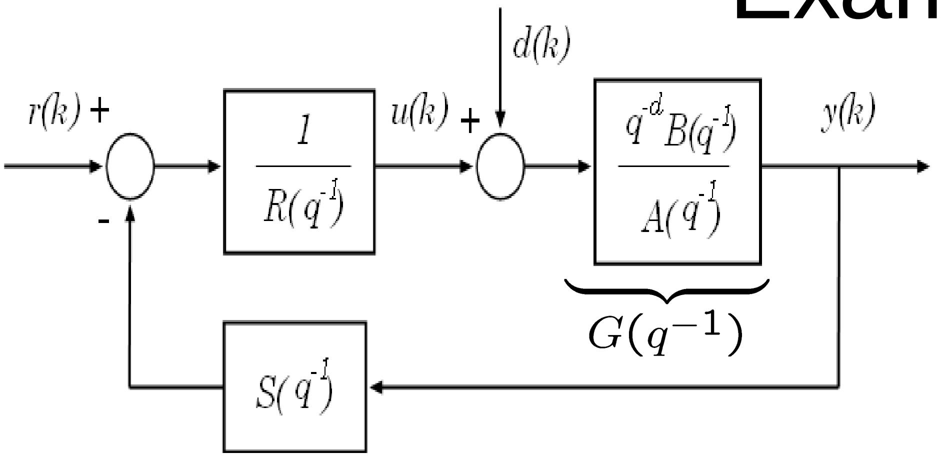
$$d(k) = d(k-1)$$

Control: $$u(k) = \frac{1}{R(q^{-1})}\Big[r(k) - S(q^{-1})y(k)\Big]$$

$$R(q^{-1}) = 1 - 0.1q^{-1} - 0.33q^{-2} - 0.57q^{-3}$$

$$S(q^{-1}) = 0.31 - 0.28q^{-1}$$

Example



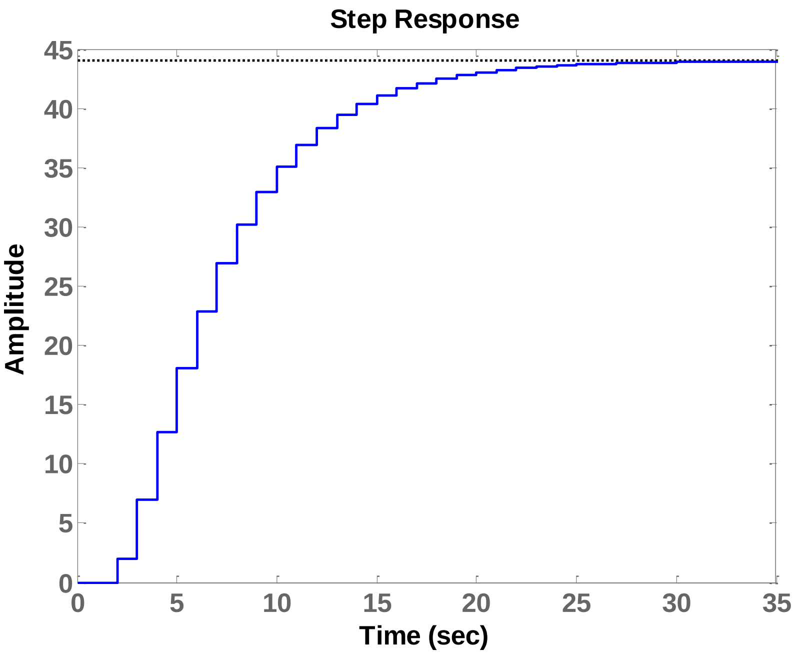
$$G(q^{-1}) = \frac{q^{-2}(2 + 2.4q^{-1})}{(1 - 1.2q^{-1})}$$

Closed-loop dynamics:

$$y(k) = \frac{q^{-\mathrm{d}} B^u(q^{-1})}{A_c'(q^{-1})} r(k)$$

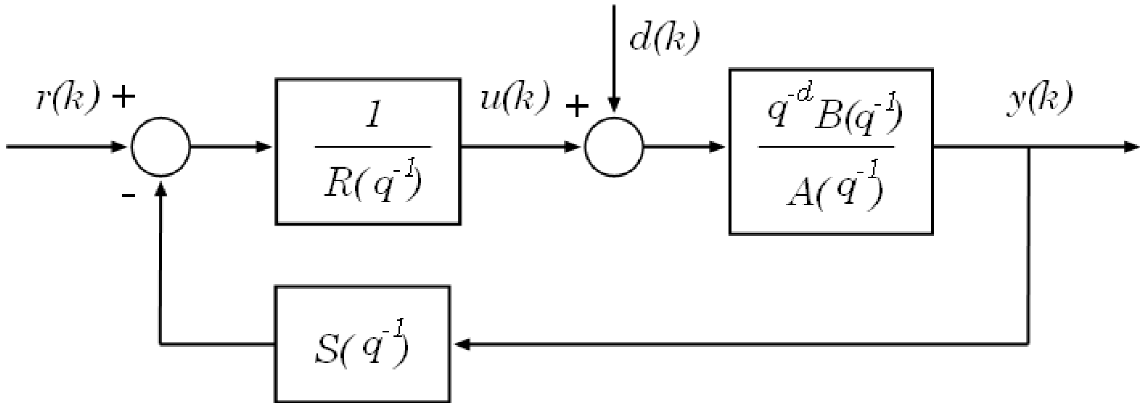
$$y(k) = \frac{q^{-2}(2 + 2.4q^{-1})}{(1 - 0.8q^{-1})(1 - 0.5q^{-1})} r(k)$$

Unit step response



Feedback Control Law

The feedback control action:



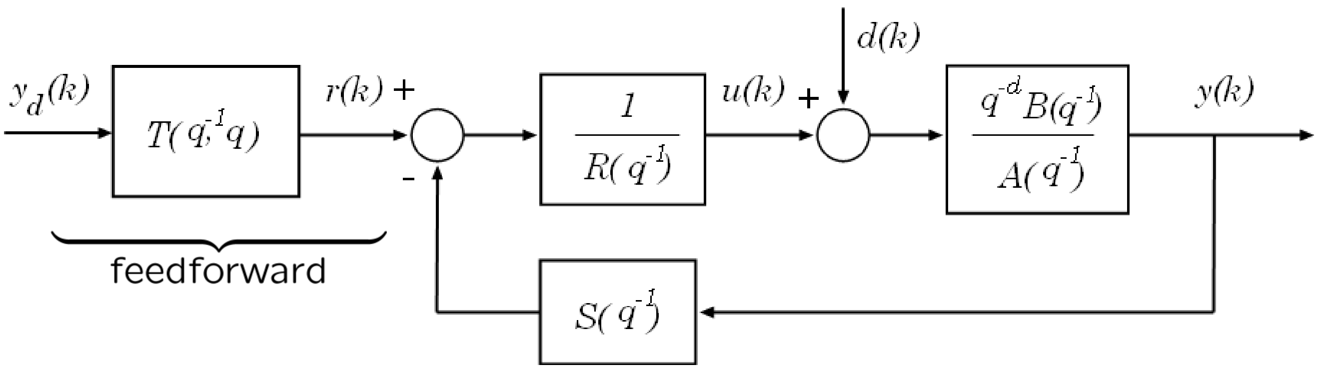
Results in the following closed-loop input/output dynamics:

$$u(k) = \frac{A(q^{-1})}{\underbrace{B^s(q^{-1})}_{\text{well-damped zeros}} A_c'(q^{-1})}\, r(k)$$

$$+ \frac{q^{-\mathrm{d}} B^u(q^{-1}) S(q^{-1})}{A_c'(q^{-1})}\, d(k)$$

Feedforward Control

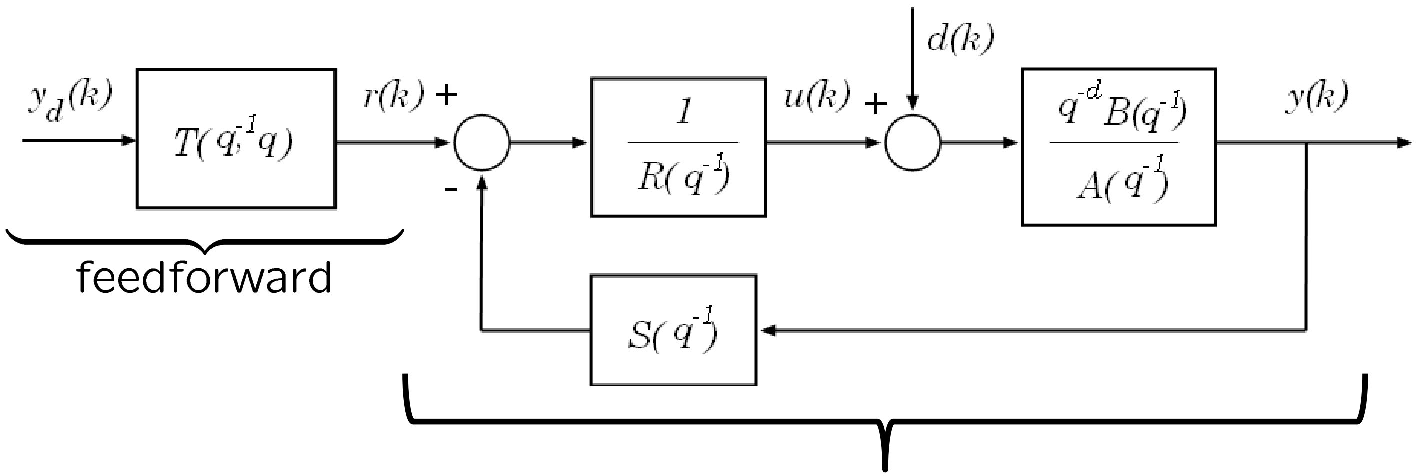
Feedforward control objective is to make $y(k)$ follow $y_d(k)$ as closely as possible.



Goal: $y(k) = y_d(k)$ or $y(k) \approx y_d(k)$

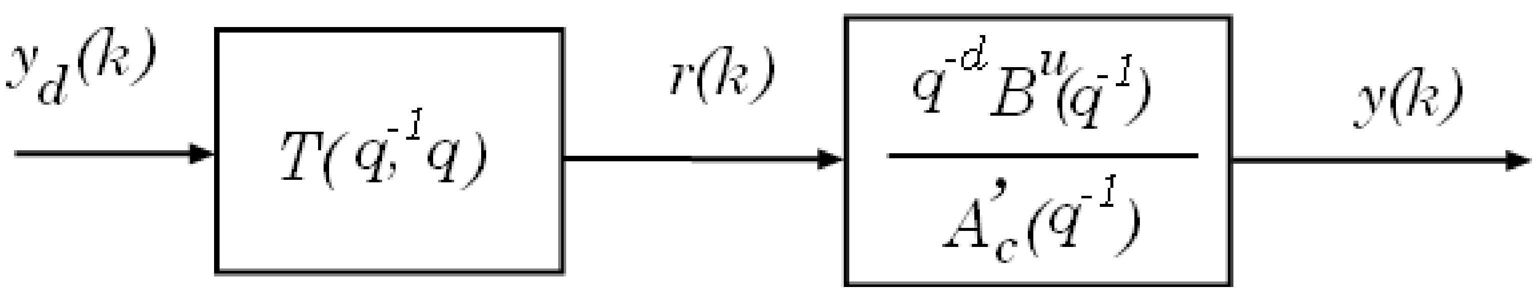
how well the objective met depends on whether the plant has unstable zeros or not

Feedforward Control Synthesis

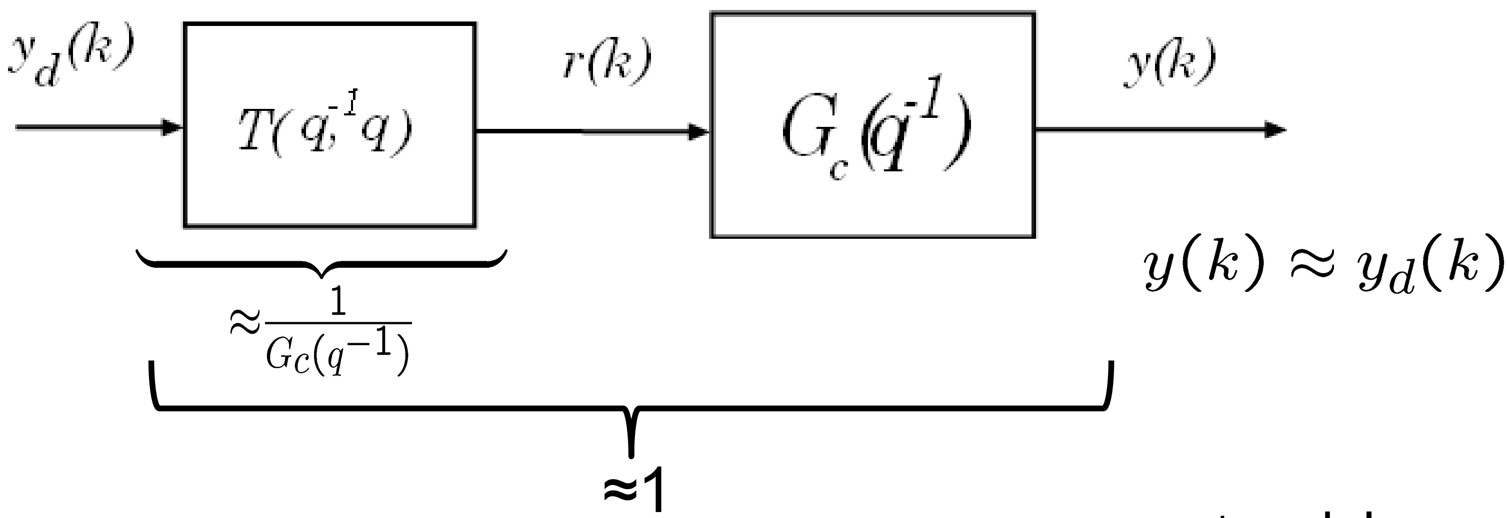


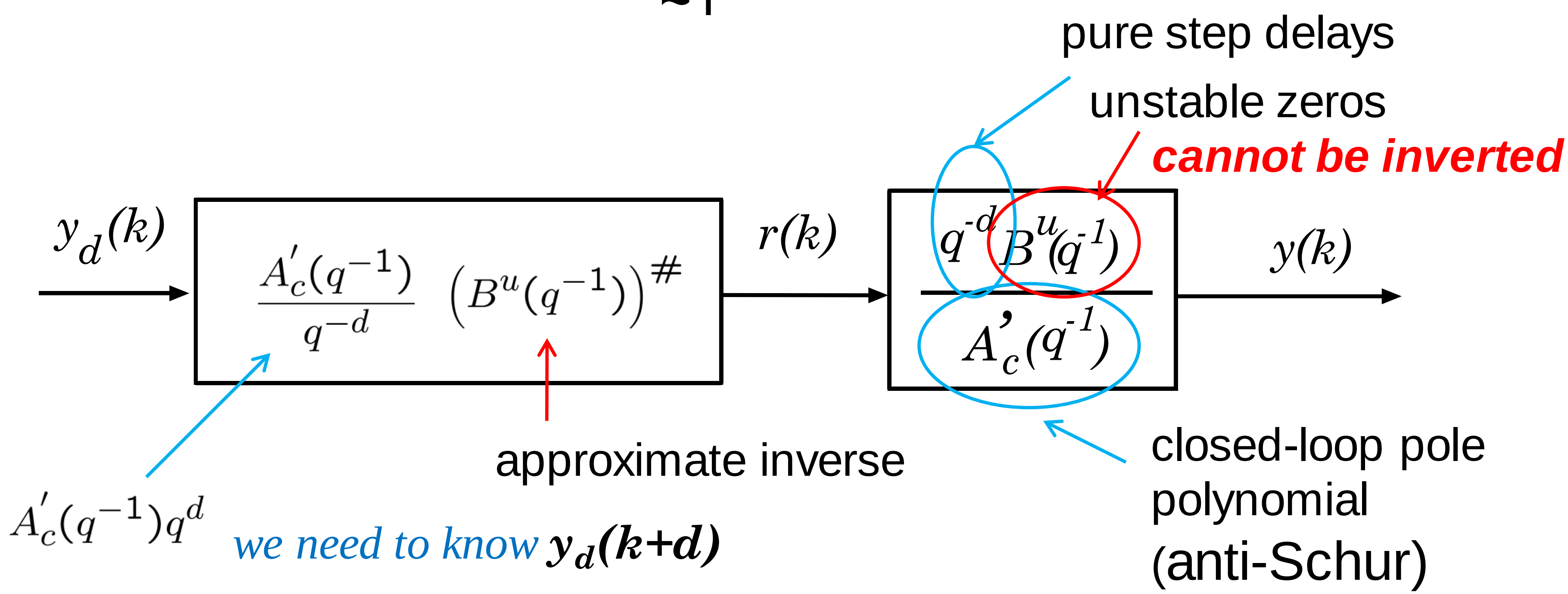
$$\frac{q^{-\mathrm{d}}B^{u}(q^{-1})}{A'_{c}(q^{-1})}$$

← closed loop dynamics from *r(k)* *to y(k)*



Feedforward control principle: **plant inversion**



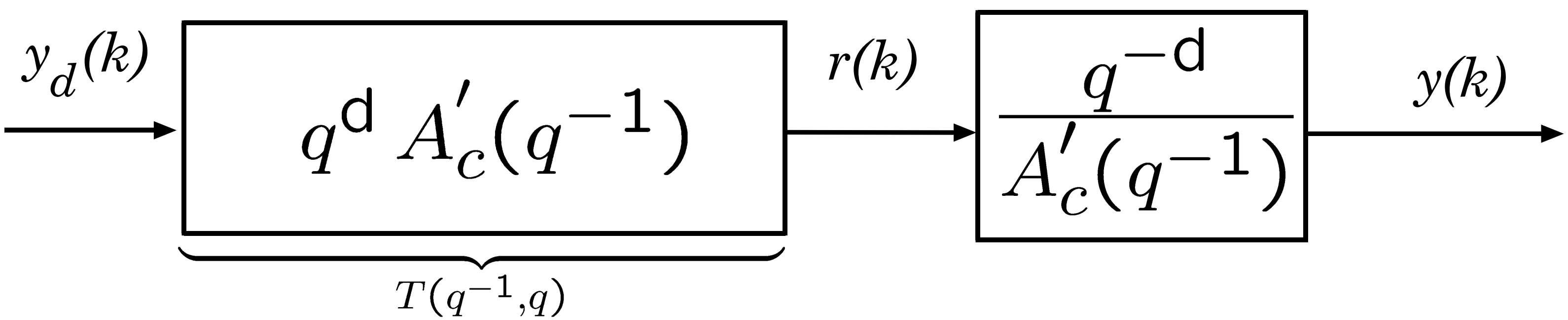


Perfect Tracking Feedforward Control

Perfect tracking can be achieved if all plant zeros are cancelable, e.g.

$$B^u(q^{-1}) = 1$$

in this case



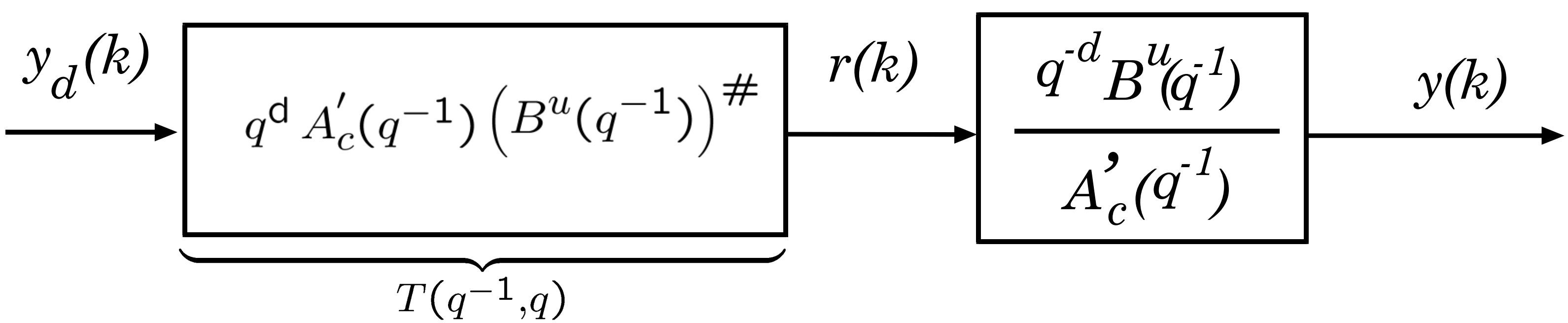
$$r(k) = A_c'(q^{-1})\, y_d(k + \mathsf{d})$$

Tracking with unstable zeros

- When the plant has unstable zeros we need to find an approximate inverse

$$\left(B^u(q^{-1})\right)^{\#}$$

$$B^u(q^{-1})\ \left(B^u(q^{-1})\right)^{\#} \approx 1$$



A-causal Bounded-Input Bounded-Output (BIBO) realization of a purely unstable operator

Let $B^u(p^{-1}) = 0 \iff |p| > 1$

i.e. all zeros of $\bar{B}^u(q) = q^{m_u} B^u(q^{-1})$ are <u>outside the unit circle</u>

Then we can interpret $\dfrac{1}{B^u(q^{-1})}$ in two ways:

- $\dfrac{1}{B^u(q^{-1})}$ is <u>**causal**</u> but unstable

- $\dfrac{1}{B^u(q^{-1})}$ is <u>**a-causal**</u> but BIBO

A-causal Bounded-Input Bounded-Output (BIBO) realization of a purely unstable operator

Example: $B^u(q^{-1}) = (2 + 2.4q^{-1}) = 2.4(0.8\bar{3} + q^{-1})$

$$\frac{1}{B^u(q^{-1})} = \frac{0.41\bar{6}}{0.8\bar{3} + q^{-1}} \longleftarrow$$ *unstable causal operator*

Using an infinite series expansion,

$$\frac{0.41\bar{6}}{(0.8\bar{3} + q^{-1})} = \frac{0.41\bar{6}q}{0.8\bar{3}q + 1}$$

$$= \underbrace{0.41\bar{6}q\left[1 - 0.8\bar{3}q + (0.8\bar{3}q)^2 - (0.8\bar{3}8q)^3 \cdots + \cdots (-1)^n (0.8\bar{3}q)^n\right]}_{\textit{infinite dimensional a-causal operator}}$$

A-causal BIBO realization of a purely unstable operator

Thus,

$$y(k) = \frac{1}{2 + 2.4q^{-1}}\, u(k)$$

Can be realized either as:

$$y(k) = -1.2\, y(k-1) + 0.5\, u(k) \qquad \text{(unstable)}$$

or

$$\begin{aligned} y(k) \;&=\; 0.41\bar{6}\Big[u(k+1) - 0.8\bar{3}\, u(k+2) + (0.8\bar{3})^2\, u(k+3) \\ &\quad -(0.8\bar{3})^3\, u(k+4) + \cdots (-0.8\bar{3})^n\, u(k+n+1) + \cdots\Big] \end{aligned}$$

(a-causal BIBO)

A-causal BIBO **approximation** of a purely unstable operator

We will now describe two methods of approximating a purely unstable operator:

1) Truncated a-casual series expansion:

$$\left(B^u(q^{-1})\right)^{\#} = \beta_1 q + \beta_2 q^2 + \cdots + \beta_3 q^M$$

2) Zero-phase error feedforward operator:
(developed by Prof. Tomizuka)

Not q^{-1}

$$\left(B^u(q^{-1})\right)^{\#} = \frac{1}{[B^u(1)]^2} B^u(q)$$

Example: realizing $\left(B^u(q^{-1})\right)^{\#}$

Let, $\quad B^u(q^{-1}) = (2 + 2.4q^{-1})$

1) Truncated a-casual series expansion:

$$\left(B^u(q^{-1})\right)^{\#} = 0.41\bar{6}\left[q - 0.8\bar{3}q^2 + (0.8\bar{3}q)^3 - (0.8\bar{3}8q)^4\right]$$

2) Zero-phase error feedforward operator:

Not q^{-1}

$$\left(B^u(q^{-1})\right)^{\#} = \frac{1}{[4.4]^2}(2 + 2.4q)$$

Zero-phase error tracking

One of the most popular feedforward techniques for systems with unstable zeros.

$$\left(B^u(q^{-1})\right)^{\#} = \frac{1}{|B^u(1)|^2} B^u(q)$$

Define the zero-phase operator

$$G_{zp}(q^{-1}, q) = B^u(q^{-1}) \left(B^u(q^{-1})\right)^{\#}$$

$$= \frac{B^u(q^{-1})\, B^u(q)}{[B^u(1)]^2}$$

Zero-phase error transfer function

A-causal zero-phase transfer function:

$$G_{zp}(z^{-1}, z) = \frac{B^u(z^{-1})\, B^u(z)}{[B^u(1)]^2}$$

Properties:

- It has zero-phase, i.e. $\mathrm{Im}\left\{G_{zp}(e^{-j\omega}, e^{j\omega})\right\} = 0$
- It has unity dc gain, i.e. $G_{zp}(e^{-0}, e^{0}) = 1$

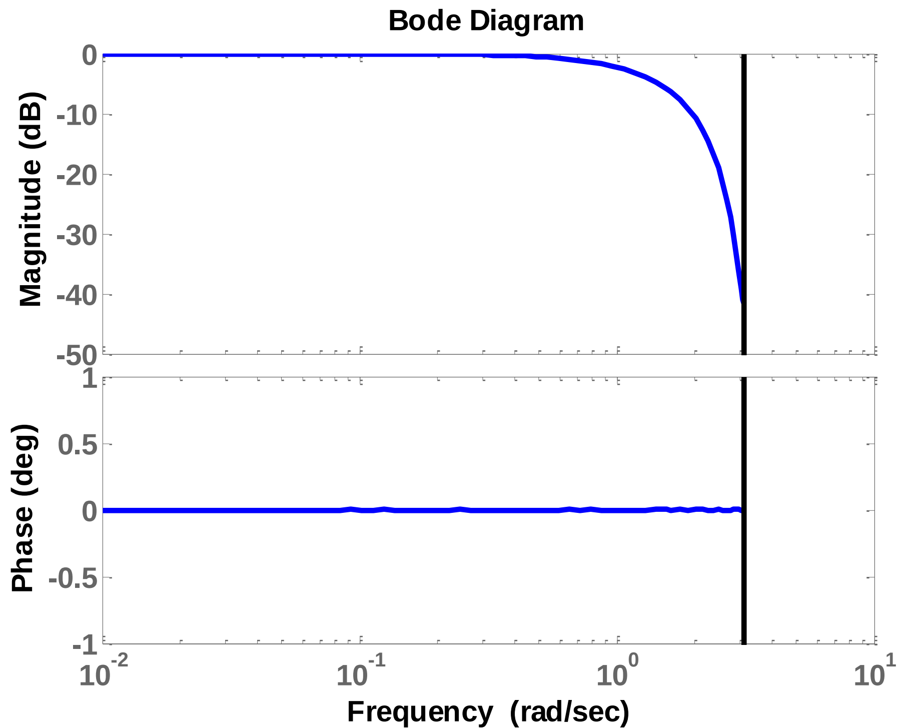
Example: realizing $\left(B^u(q^{-1})\right)^{\#}$

Let, $B^u(q^{-1}) = (2 + 2.4q^{-1})$

- Zero-phase feedforward: $\left(B^u(q^{-1})\right)^{\#} = \frac{1}{[4.4]^2}(2 + 2.4q)$

- Zero-phase transfer function:

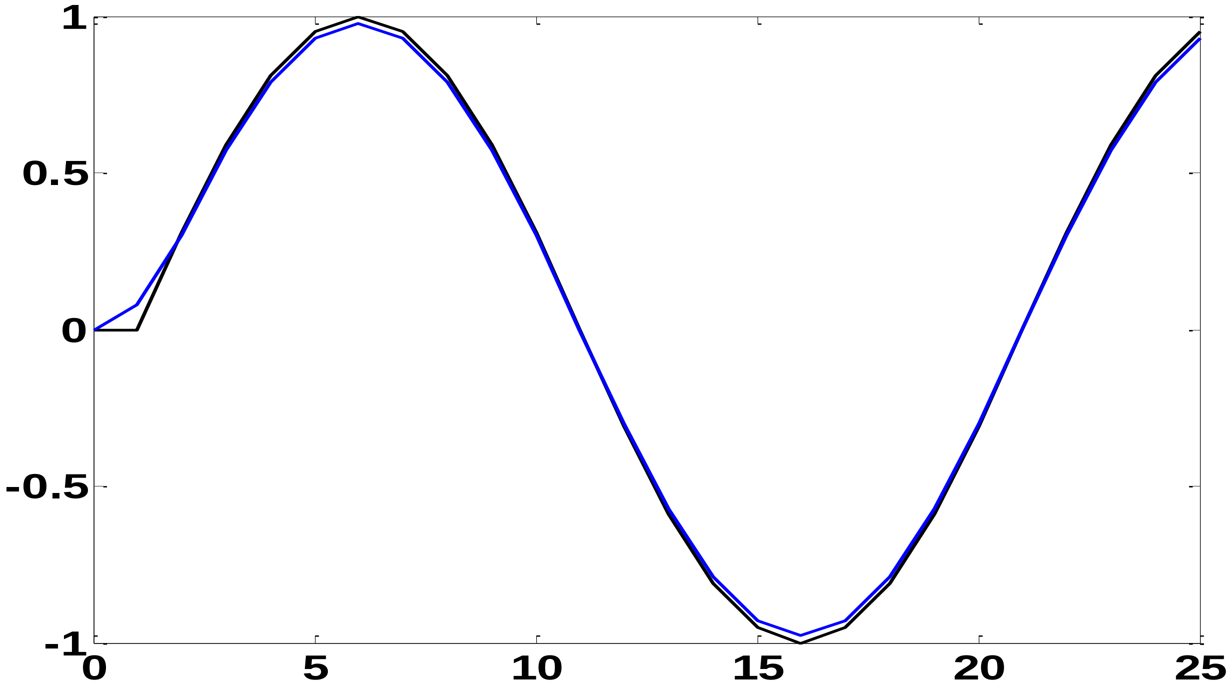
$$G_{zp}(z^{-1}, z) = \frac{(2 + 2.4z^{-1})\,(2z + 2.4)}{[4.4]^2}$$



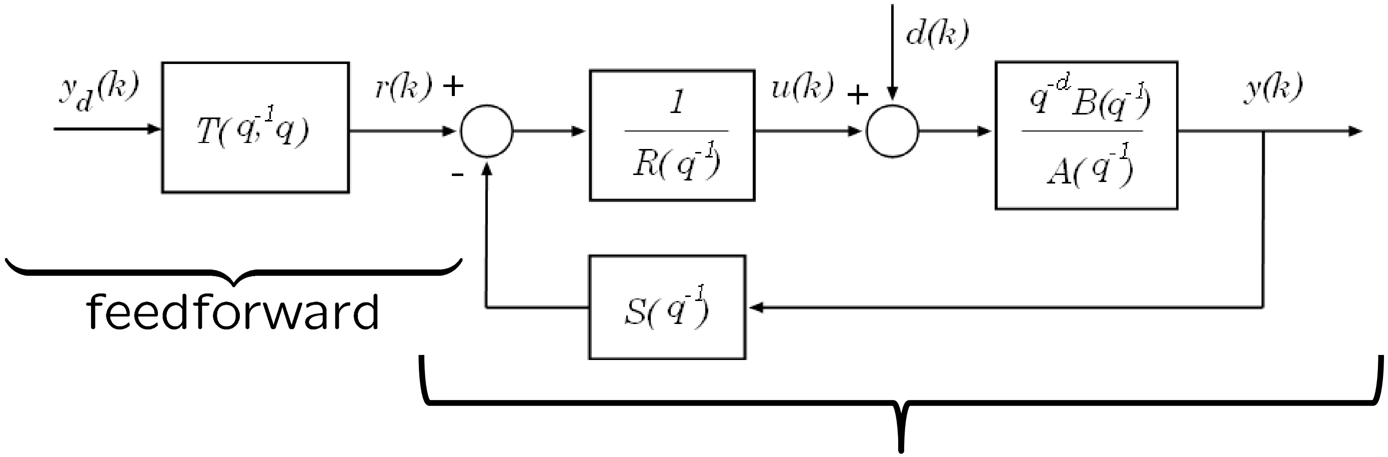
Sinusoidal zero-phase error tracking

If $y_d(k)$ is a sinusoidal, there will be no phase shift between $y_d(k)$ and $y(k)$

$$y(k) = \frac{B^u(q^{-1})\,B^u(q)}{[B^u(1)]^2}\,y_d(k)$$



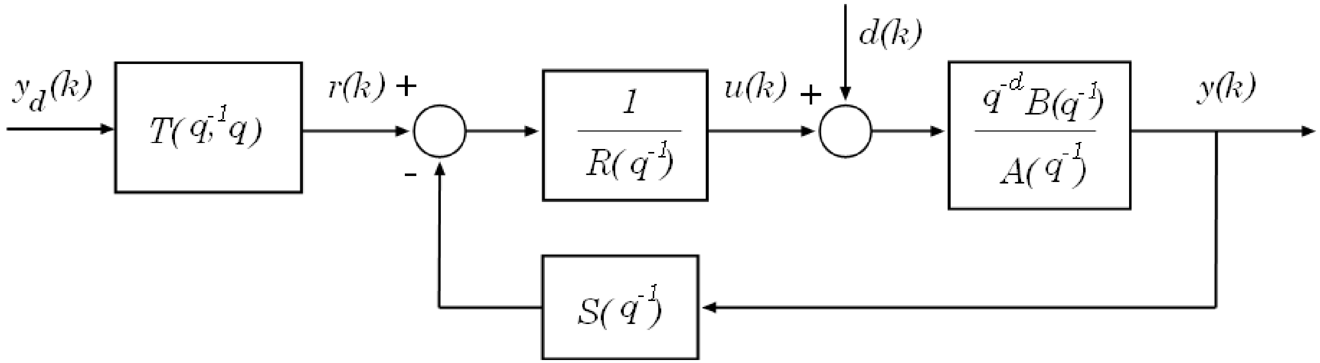
Zero-phase error feedforward



$$\frac{q^{-\mathrm{d}} B^u(q^{-1})}{A_c'(q^{-1})}$$

$$\boxed{T(q^{-1}, q) = A_c'(q^{-1})\, q^{\mathrm{d}}\, \frac{B^u(q)}{[B^u(1)]^2}}$$

Zero-phase error feedforward



Closed-loop dynamics:

$$y(k) = \frac{B^u(q^{-1})\,B^u(q)}{[B^u(1)]^2}\,y_d(k)$$