ME 233 Advanced Control II

Lecture 13

Frequency-Shaped
Linear Quadratic Regulator

(ME233 Class Notes pp.FSLQ1-FSLQ5)

Outline

- Parseval’s theorem
- Frequency-shaped LQR
  - Implementation
- Frequency-shaped LQR with reference input

Infinite-Horizon LQR (review)

nth order LTI system:

$$x(k+1) = Ax(k) + Bu(k) \qquad\qquad x(0) = x_o$$

Find the optimal control:

$$u(k) = -Kx(k)$$

which minimizes the cost functional:

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k) Q x(k) + u^T(k) R u(k) \right\}$$

$$Q = Q^T \succeq 0 \qquad\qquad R = R^T \succ 0$$

Parseval's theorem

- Let *f(k)* be a map from the integers to R^n

- Its (symmetric) Fourier transform is defined by

$$F(e^{j\omega}) = \mathcal{F}(f(k)) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} f(k) e^{-j\omega k}$$

and

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(e^{j\omega}) e^{+j\omega k} d\omega$$

Parseval's theorem

$$\sum_{k=-\infty}^{\infty} f^T(k) f(k) = \int_{-\pi}^{\pi} F^*(e^{j\omega}) F(e^{j\omega}) d\omega$$

where

$$F(e^{j\omega}) = \mathcal{F}(f(k))$$

$F^*(e^{j\omega}) = F^T(e^{-j\omega})$ (complex conjugate transpose)

$$\sum_{k=-\infty}^{\infty} f^T(k) f(k) = \int_{-\pi}^{\pi} F^*(e^{j\omega}) F(e^{j\omega}) d\omega$$

Proof:

$$\sum_{k=-\infty}^{\infty} f^T(k) f(k) = \sum_{k=-\infty}^{\infty} f^T(k) \overbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(e^{j\omega}) e^{+j\omega k} d\omega \right)}^{f(k)}$$

$$= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} f^T(k) \frac{1}{\sqrt{2\pi}} F(e^{j\omega}) e^{+j\omega k} \right) d\omega$$

$$= \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} f^T(k) e^{+j\omega k} \right)}_{F^T(e^{-j\omega})} F(e^{j\omega}) d\omega$$

$\blacksquare$

Frequency Cost Function

By Parseval's theorem, the cost function:

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k) Q x(k) + u^T(k) R u(k) \right\}$$

with $\begin{cases} x(k) = 0 & k < 0 \\ u(k) = 0 & k < 0 \end{cases}$

is equivalent to the cost function

$$J = \int_{-\pi}^{\pi} \left\{ X^*(e^{j\omega}) Q X(e^{j\omega}) + U^*(e^{j\omega}) R U(e^{j\omega}) \right\} d\omega$$

$$X(e^{j\omega}) = \mathcal{F}(x(k)) \qquad U(e^{j\omega}) = \mathcal{F}(u(k))$$

Frequency-Shaped Cost Function

Key idea: Make matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ functions of frequency

$$J = \int_{-\pi}^{\pi} \Big\{ X^*(e^{j\omega}) \underline{Q(e^{j\omega})} X(e^{j\omega}) + U^*(e^{j\omega}) \underline{R(e^{j\omega})} U(e^{j\omega}) \Big\} \, d\omega$$

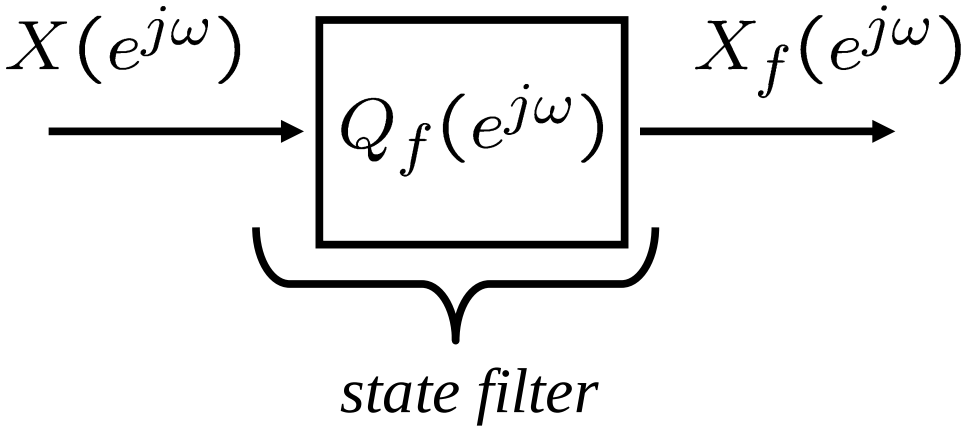
where

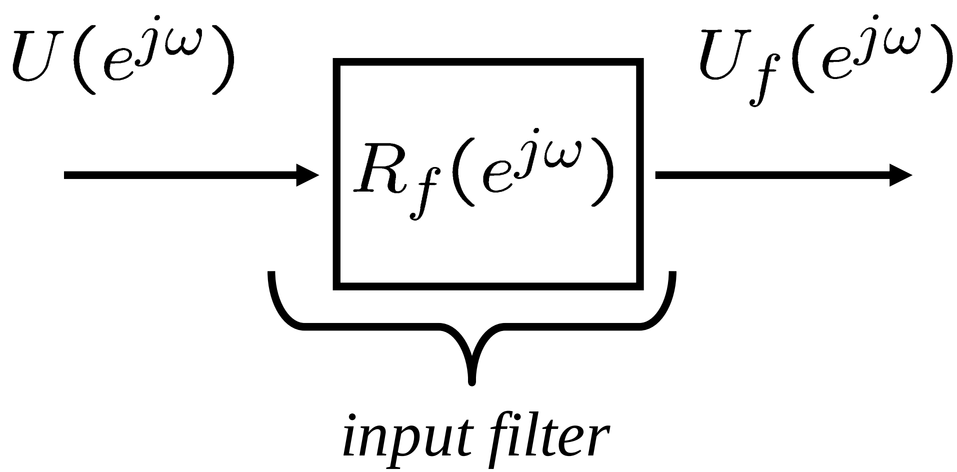
$$\underline{Q(e^{j\omega})} = Q_f^*(e^{j\omega}) Q_f(e^{j\omega}) \succeq 0$$

$$\underline{R(e^{j\omega})} = R_f^*(e^{j\omega}) R_f(e^{j\omega}) \succ 0$$

Frequency-Shaped Cost Function

Define the state and input filters





Frequency-Shaped Cost Function

$$J = \int_{-\pi}^{\pi} \Big\{ X^*(e^{j\omega}) \overbrace{Q(e^{j\omega})}^{Q_f^*(e^{j\omega})Q_f(e^{j\omega})} X(e^{j\omega})$$

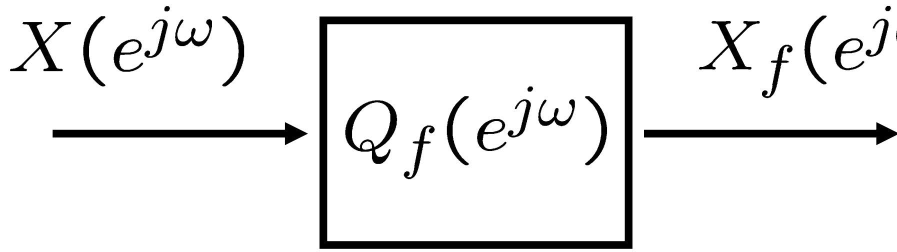
$$+ U^*(e^{j\omega}) \underbrace{R(e^{j\omega})}_{R_f^*(e^{j\omega})R_f(e^{j\omega})} U(e^{j\omega}) \Big\} \, d\omega$$

can be written

$$J = \int_{-\pi}^{\pi} \Big\{ X_f^*(e^{j\omega}) X_f(e^{j\omega}) + U_f^*(e^{j\omega}) U_f(e^{j\omega}) \Big\} \, d\omega$$

Realizing the filters using LTI's

Let



be realized by

$$z_1(k+1) = A_1 z_1(k) + B_1 x(k)$$

$$x_f(k) = C_1 z_1(k) + D_1 x(k)$$

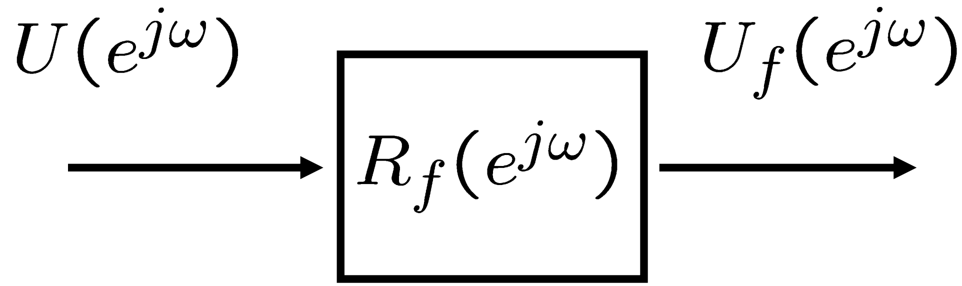
so that

$$Q_f(z) = C_1(zI - A_1)^{-1} B_1 + D_1$$

is causal or strictly causal.

Realizing the filters using LTI's

Let



be realized by

$$z_2(k+1) = A_2 z_2(k) + B_2 u(k)$$
$$u_f(k) = C_2 z_2(k) + D_2 u(k)$$

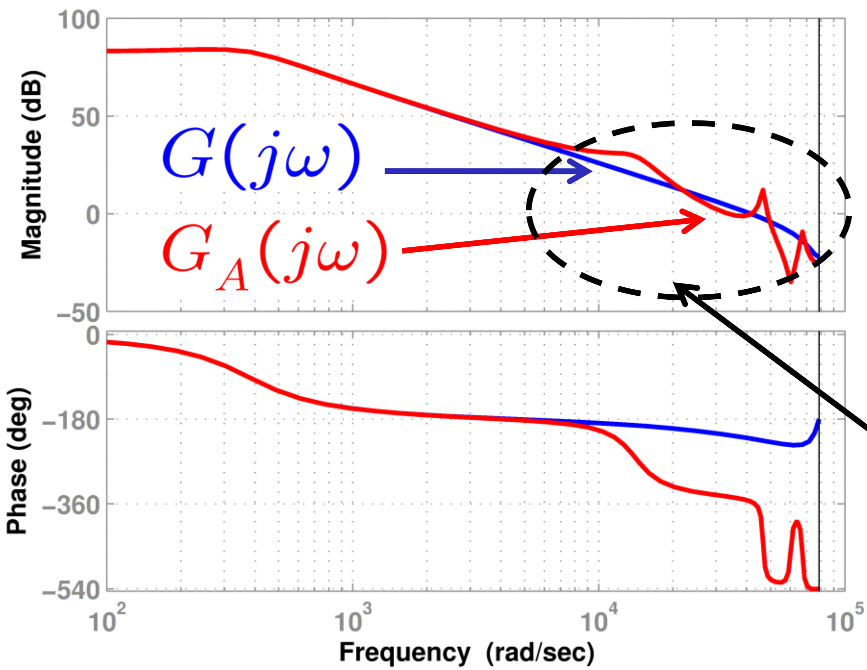
(with $D_2^T D_2 \succ 0$) so that

$$R_f(z) = C_2(zI - A_2)^{-1} B_2 + D_2$$

is causal (but not strictly causal)

Example: Hard Disk Drive

Consider a simplified model of a voice coil motor and suspension (from control input *u(k)* to read/write head position *y(k)*)



$$G_A(z) = G(z)\,[1 + \Delta(z)]$$

uncertainty

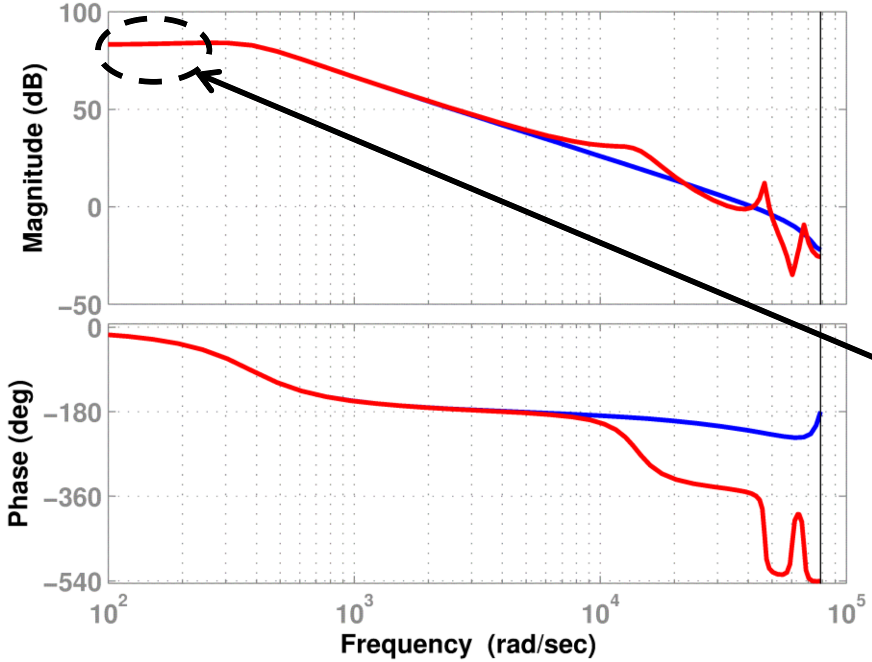
nominal model

actual plant

high-frequency resonance modes are neglected in the nominal model

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}\right\} \; G(z)$$

Example: Frequency State Weight Q(jω)



$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

we want zero steady state (i.e. dc) error under a step input

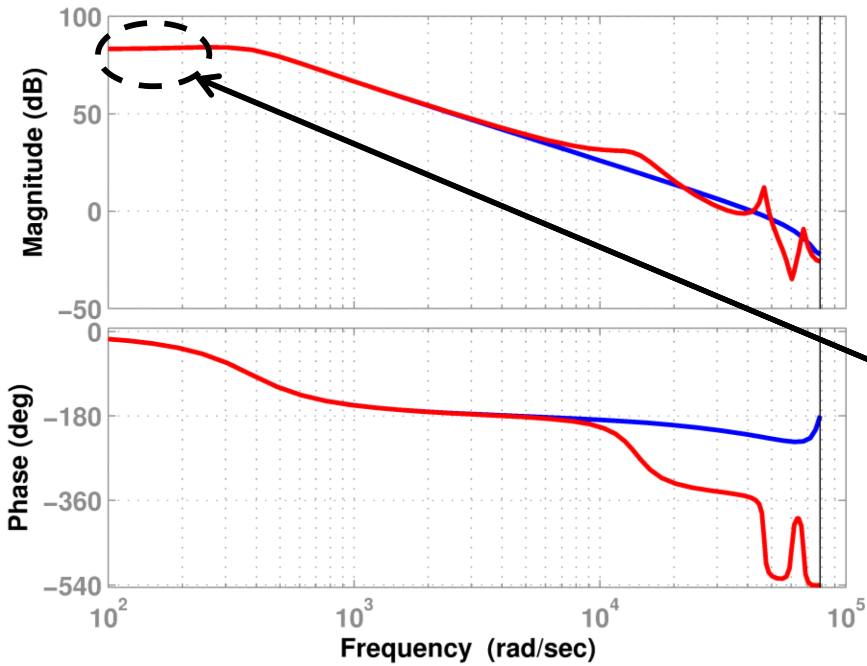
$\Rightarrow$ set position cost function weight to go to ∞ as $\omega \to 0$

Example

Set weight on $|Y(e^{j\omega})|^2$ to $\left|\frac{1}{e^{j\omega}-1}\right|^2$

$$\Rightarrow \underbrace{X^*(e^{j\omega})C^T}_{Y^*(e^{j\omega})} \overbrace{\left|\frac{1}{e^{j\omega}-1}\right|^2}^{Q(e^{j\omega})} \underbrace{CX(e^{j\omega})}_{Y(e^{j\omega})}$$

Example: Frequency State Weight Q(jω)



$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

we want zero steady state (i.e. dc) error under a step input

set position cost function weight to go to ∞ as $\omega \rightarrow 0$

Example

$$X^*(e^{j\omega})C^T\left|\frac{1}{e^{j\omega}-1}\right|^2 CX(e^{j\omega}) = \underbrace{X^*(e^{j\omega})C^T\overbrace{\left(\frac{1}{e^{-j\omega}-1}\right)}^{Q_f^*(e^{j\omega})}}_{X_f^*(e^{j\omega})}\underbrace{\overbrace{\left(\frac{1}{e^{j\omega}-1}\right)}^{Q_f(e^{j\omega})}CX(e^{j\omega})}_{X_f(e^{j\omega})}$$

Example: Frequency State Weight Q(jω)

$$X(e^{j\omega}) \longrightarrow \boxed{Q_f(e^{j\omega})} \longrightarrow X_f(e^{j\omega})$$

state space realization

$$z_1(k+1) = A_1 z_1(k) + B_1 x(k)$$

$$x_f(k) = C_1 z_1(k) + D_1 x(k)$$

Example

$$X_f(e^{j\omega}) = \underbrace{\frac{1}{e^{j\omega}-1}C}_{Q_f(e^{j\omega})} X(e^{j\omega})$$

state space realization

$$Q_f(z) = \frac{1}{z-1}C \quad \Rightarrow \quad \begin{cases} z_1(k+1) = \underset{\substack{\uparrow \\ A_1}}{1}\, z_1(k) + \underset{\substack{\uparrow \\ B_1}}{C}\, x(k) \\[2em] x_f(k) = \underset{\substack{\uparrow \\ C_1}}{1}\, z_1(k) + \underset{\substack{\uparrow \\ D_1}}{0}\, x(k) \end{cases}$$

Example: Hard Disk Drive

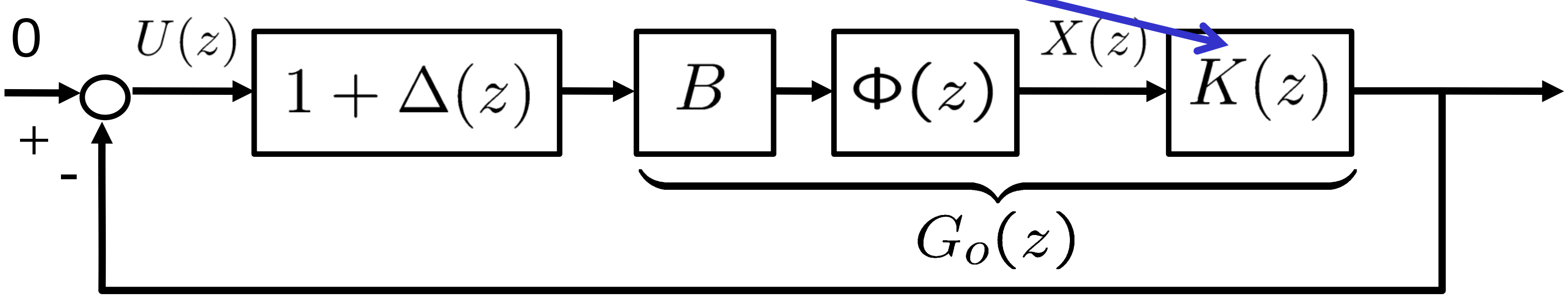
$$J = \int_{-\pi}^{\pi} \left\{ X^*(e^{j\omega}) Q(e^{j\omega}) X(e^{j\omega}) + \rho U^*(e^{j\omega}) U(e^{j\omega}) \right\} d\omega$$

Apply control design to nominal model

Weights: $Q(e^{j\omega}) = C^T \left| \frac{1}{e^{j\omega} - 1} \right|^2 C$

$\rho = 10^6$

FS-LQR is a <u>dynamic</u> state feedback



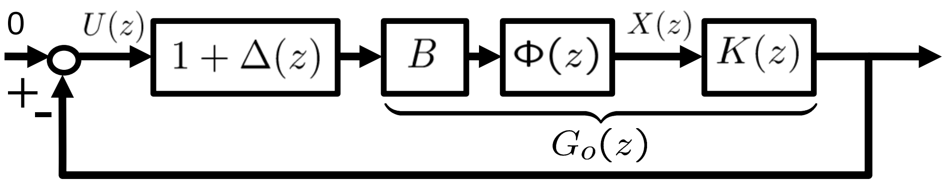
Sufficient condition for robustness (by small gain theorem) :

$$|T(e^{j\omega})| \leq \frac{1}{|\Delta(e^{j\omega})|}$$

$$T(z) = \frac{G_o(z)}{1 + G_o(z)}$$

Example: Hard Disk Drive

$$J = \int_{-\pi}^{\pi} \left\{ X^*(e^{j\omega}) Q(e^{j\omega}) X(e^{j\omega}) + \rho U^*(e^{j\omega}) U(e^{j\omega}) \right\} d\omega$$



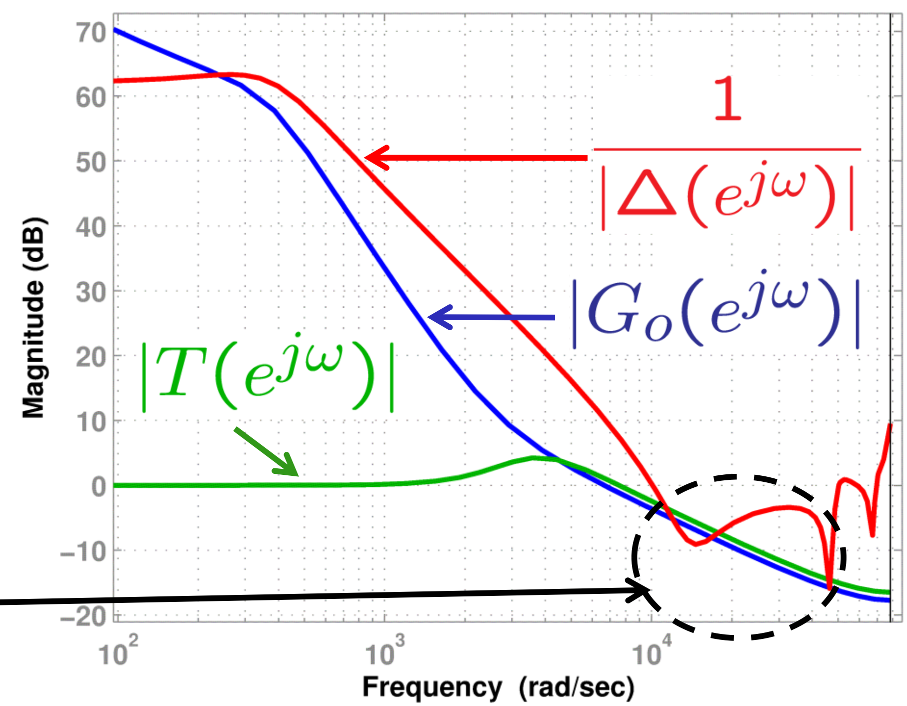
Weights: $Q(e^{j\omega}) = C^T \left| \frac{1}{e^{j\omega}-1} \right|^2 C$

$$\rho = 10^6$$

$$T(z) = \frac{G_o(z)}{1 + G_o(z)}$$

sufficient condition for robustness
(by small gain theorem) :

$$|T(e^{j\omega})| \leq \frac{1}{|\Delta(e^{j\omega})|}$$



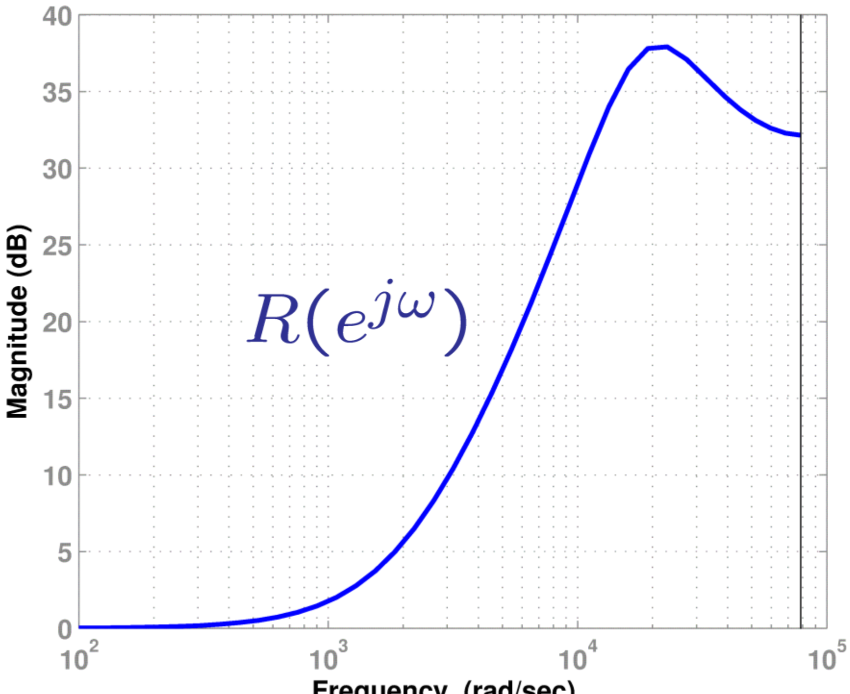
potential
lack of robustness

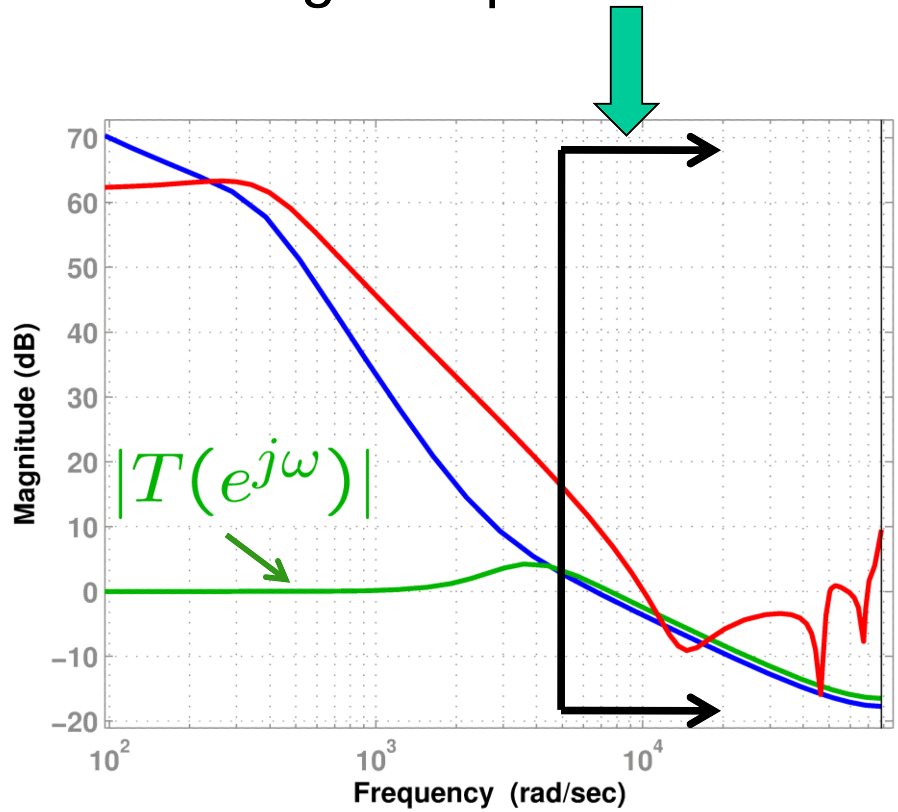
Example: Frequency Control Weight R(jω)

$$J = \int_{-\pi}^{\pi} \left\{ X^*(e^{j\omega}) Q(e^{j\omega}) X(e^{j\omega}) + \rho U^*(e^{j\omega}) \underbrace{R(e^{j\omega})} U(e^{j\omega}) \right\} d\omega$$

increase control penalty at high-frequencies

Example

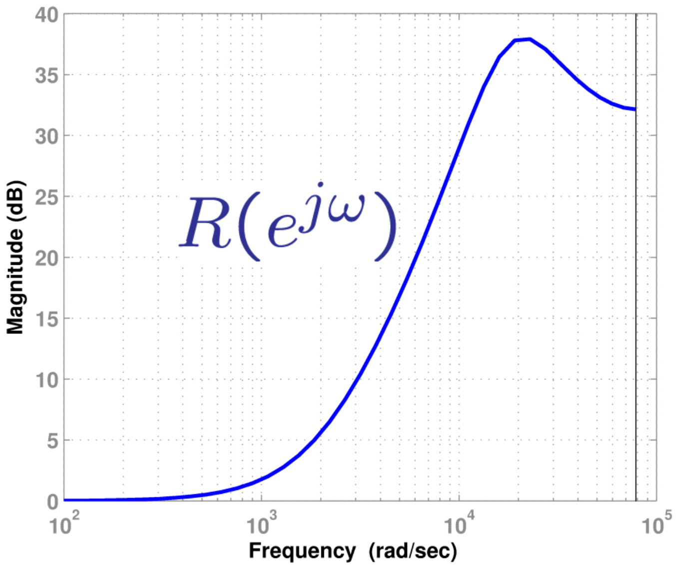




Example: Frequency Control Weight R(jω)

$$J = \int_{-\pi}^{\pi} \left\{ X^*(e^{j\omega}) Q(e^{j\omega}) X(e^{j\omega}) + \underbrace{\rho U^*(e^{j\omega}) R(e^{j\omega}) U(e^{j\omega})}_{U_f^*(e^{j\omega}) U_f(e^{j\omega})} \right\} d\omega$$

Example



$$\hat{R}_f(s) = \alpha^2 \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{s^2 + 2\bar{\zeta}(\alpha\omega_n)s + (\alpha\omega_n)^2}$$

$\alpha = 2.5$

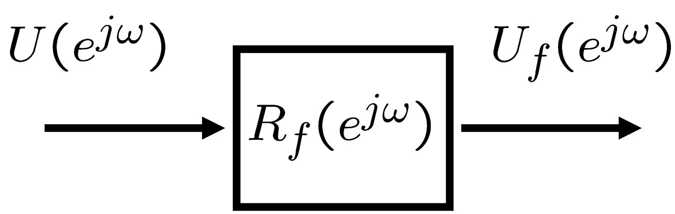
$\zeta = 1.5$

$\omega_n = 7500$

$\bar{\zeta} = 0.6$

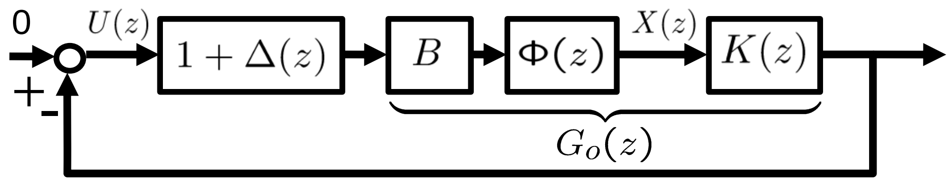
Discretize using ZOH

$R_f(z)$

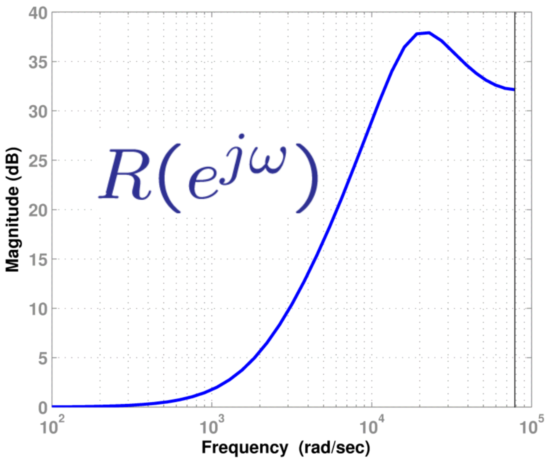


Example: Frequency Control Weight R(jω)

$$J = \int_{-\pi}^{\pi} \left\{ X^*(e^{j\omega}) Q(e^{j\omega}) X(e^{j\omega}) + \rho U^*(e^{j\omega}) R(e^{j\omega}) U(e^{j\omega}) \right\} d\omega$$

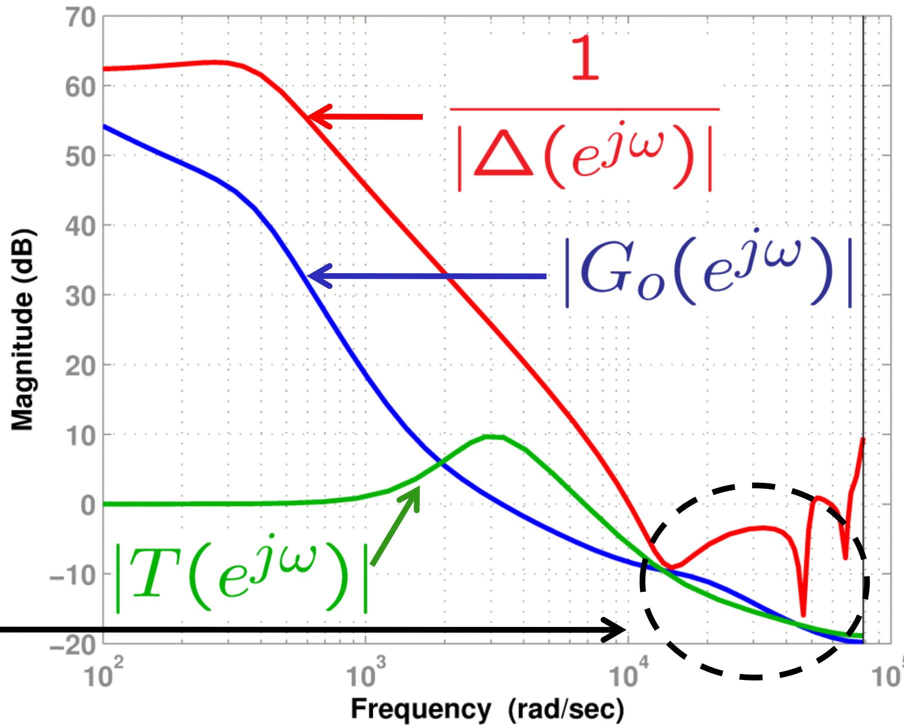


Example



$$Q(e^{j\omega}) = C^T \left| \frac{1}{e^{j\omega}-1} \right|^2 C$$

$$\rho = 10^6$$



Robustness condition
is satisfied

Cost Function Realization

$$J = \int_{-\pi}^{\pi} \Big\{ X^*(e^{j\omega}) Q(e^{j\omega}) X(e^{j\omega}) + U^*(e^{j\omega}) R(e^{j\omega}) U(e^{j\omega}) \Big\} \, d\omega$$

$$\Updownarrow$$

$$J = \int_{-\pi}^{\pi} \Big\{ X_f^*(e^{j\omega}) X_f(e^{j\omega}) + U_f^*(e^{j\omega}) U_f(e^{j\omega}) \Big\} \, d\omega$$

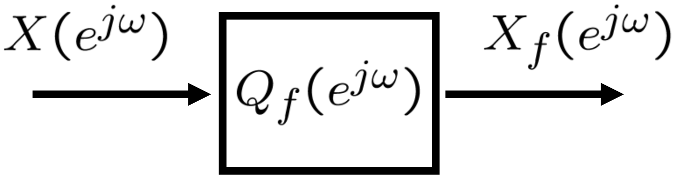
$$\Updownarrow$$

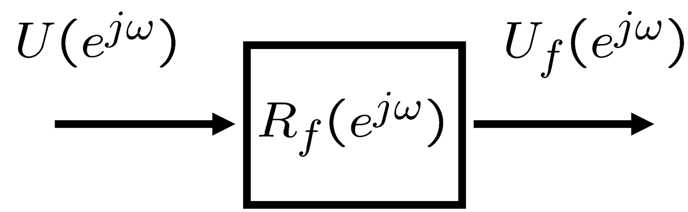
$$J = \sum_{k=0}^{\infty} \Big\{ x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \Big\}$$

Cost Function Realization

$$J = \sum_{k=0}^{\infty} \left\{ x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \right\}$$

where





state space realization

$$z_1(k+1) = A_1 z_1(k) + B_1 x(k)$$

$$x_f(k) = C_1 z_1(k) + D_1 x(k)$$

state space realization

$$z_2(k+1) = A_2 z_2(k) + B_2 u(k)$$

$$u_f(k) = C_2 z_2(k) + D_2 u(k)$$

Cost Function Realization

$$J = \sum_{k=0}^{\infty} \left\{ x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \right\}$$

$$z_1(k+1) = A_1 z_1(k) + B_1 x(k) \qquad z_2(k+1) = A_2 z_2(k) + B_2 u(k)$$

$$x_f(k) = C_1 z_1(k) + D_1 x(k) \qquad u_f(k) = C_2 z_2(k) + D_2 u(k)$$

Plus: $$x(k+1) = Ax(k) + Bu(k)$$

define extended state $$x_e(k) = \begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix}$$

Cost Function Realization

$$J = \sum_{k=0}^{\infty} \left\{ x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \right\}$$

We can combine state equations and output as follows:

$$\begin{bmatrix} x(k+1) \\ z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix} u(k)$$

$$\begin{bmatrix} x_f(k) \\ u_f(k) \end{bmatrix} = \begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix} \begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ D_2 \end{bmatrix} u(k)$$

Extended System Dynamics

$$\underbrace{\begin{bmatrix} x(k+1) \\ z_1(k+1) \\ z_2(k+1) \end{bmatrix}}_{x_e(k+1)} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ B_1 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\begin{bmatrix} B \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u(k)$$

$$x_e(k+1) = A_e x_e(k) + B_e u(k)$$

Extended System Cost

$$J = \sum_{k=0}^{\infty} \left\{ x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \right\}$$

Using

$$\begin{bmatrix} x_f(k) \\ u_f(k) \end{bmatrix} = \underbrace{\begin{bmatrix} D_1 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix}}_{C_e} \underbrace{\begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\begin{bmatrix} 0 \\ D_2 \end{bmatrix}}_{D_e} u(k)$$

the cost can be expressed

$$J = \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C_e^T \\ D_e^T \end{bmatrix} \begin{bmatrix} C_e & D_e \end{bmatrix} \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix} \right\}$$

FSLQR problem statement

Minimize

$$J = \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C_e^T \\ D_e^T \end{bmatrix} \begin{bmatrix} C_e & D_e \end{bmatrix} \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix} \right\}$$

Subject to

$$x_e(k+1) = A_e x_e(k) + B_e u(k)$$

This is a standard LQR problem!

FSLQR solution

The optimal control law is

$$u^o(k) = -K_e x_e(k)$$

$$K_e = [B_e^T P B_e + D_e^T D_e]^{-1}[B_e^T P A_e + D_e^T C_e]$$

where *P* is the solution of the DARE

$$\begin{aligned} P = {} & A_e^T P A_e + C_e^T C_e \\ & -[A_e^T P B_e + C_e^T D_e][B_e^T P B_e + D_e^T D_e]^{-1}[B_e^T P A_e + D_e^T C_e] \end{aligned}$$

for which $A_e - B_e K_e$ is Schur

FSLQR existence

The optimal control law exists if

- (A_e, B_e) stabilizable
- The state-space realization $C_e(zI-A_e)^{-1}B_e + D_e$ has no transmission zeros on the unit circle

Sufficient conditions for FSLQR

The optimal control law exists if the following hold:

1. (A,B) is stabilizable

2. Q_f and R_f are stable (i.e. A_1 and A_2 are Schur)

3. $\text{nullity}\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} = 0$ whenever $|\lambda| = 1$

4. $\text{nullity}\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} = 0$ whenever $\begin{cases} \det(A - \lambda I) = 0 \\ |\lambda| = 1 \end{cases}$

(You will be asked to show this for homework)

Remarks on existence conditions

Condition 3 from the existence conditions:

$$\text{nullity}\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} = 0 \quad \text{whenever} \quad |\lambda| = 1$$

is equivalent to the condition that

The state space realization for R_f has no transmission zeros on the unit circle

(This is because $D_2^T D_2 \succ 0$)

Remarks on existence conditions

Condition 4 from the existence conditions

$$\text{nullity}\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} = 0 \quad \text{whenever} \quad \begin{cases} \det(A - \lambda I) = 0 \\ |\lambda| = 1 \end{cases}$$

is a <u>stronger</u> version of the condition that

None of the unit circle eigenvalues of *A* are transmission zeros of the state space realization for Q_f

(The latter is not enough for FSLQR existence)

Implementation

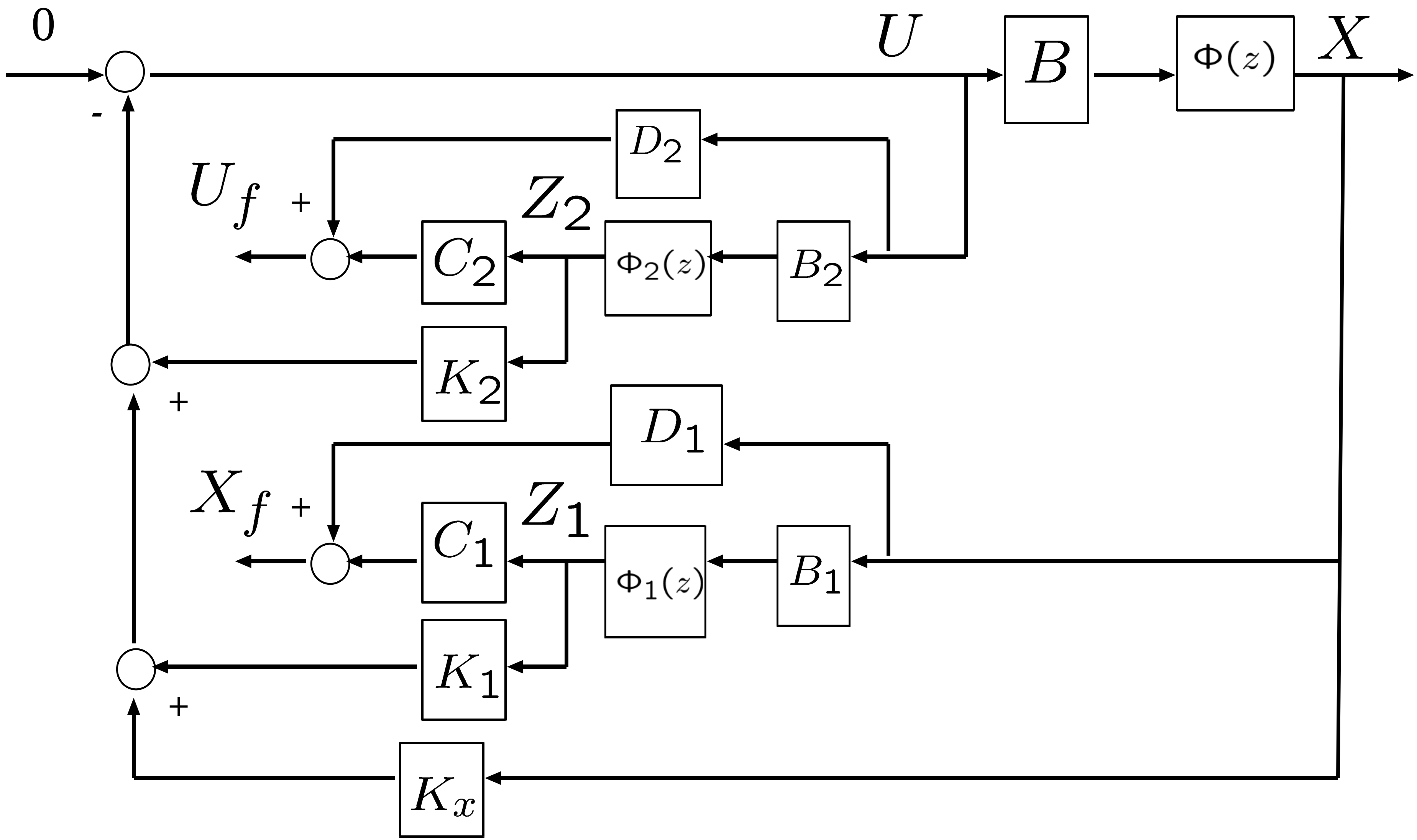
- Control

$$u(k) = -K_e x_e(k)$$

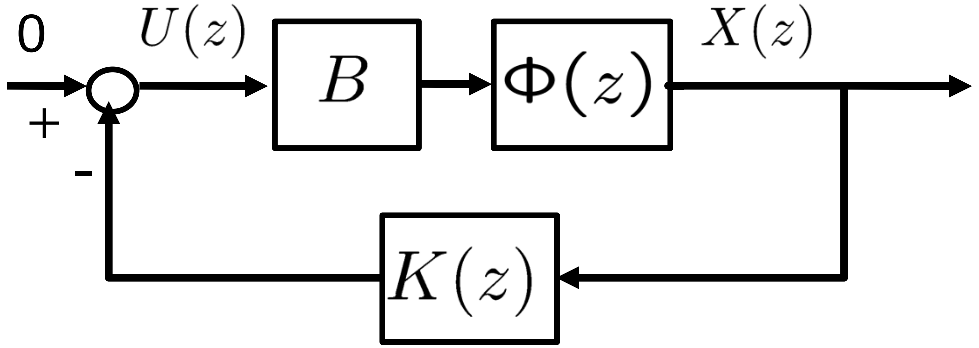
$$= -\begin{bmatrix} K_x & K_1 & K_2 \end{bmatrix} \begin{bmatrix} x(k) \\ z_1(k) \\ z_2(k) \end{bmatrix}$$

$$= -K_x x(k) - K_1 z_1(k) - K_2 z_2(k)$$

Block Diagram



Equivalent Block Diagram



$$K(z) = [I + K_2 \Phi_2(z) B_2]^{-1} \, [K_x + K_1 \Phi_1(z) B_1]$$

State-space realization for *K(z)*

$$u(k) = -K_x x(k) - K_1 z_1(k) - K_2 z_2(k)$$

$$z_1(k+1) = A_1 z_1(k) + B_1 x(k)$$

$$z_2(k+1) = A_2 z_2(k) + B_2 u(k)$$

$$= A_2 z_2(k) + B_2(-K_x x(k) - K_1 z_1(k) - K_2 z_2(k))$$

$$\boxed{\begin{aligned} \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ -B_2 K_1 & A_2 - B_2 K_2 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ -B_2 K_x \end{bmatrix} x(k) \\ -u(k) &= \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + K_x x(k) \end{aligned}}$$

FSLQR with reference input

- For simplicity, we will assume a scalar output

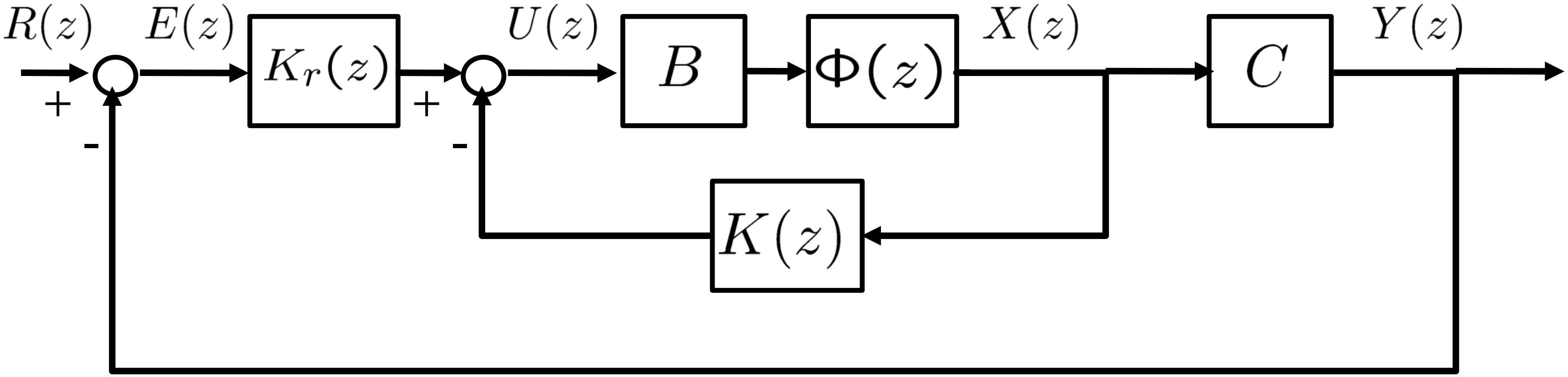
$$y(k) = Cx(k) \qquad\qquad y \in \mathcal{R}$$

- Assume that we want to design a FSLQR that will achieve asymptotic output convergence to a reference input

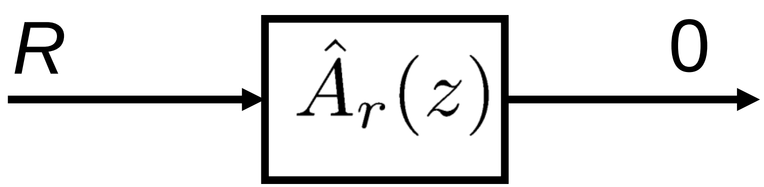
$$e(k) = r(k) - y(k)$$

$$\lim_{k \to \infty} e(k) = 0$$

FSLQR with reference input



- Assume that the reference input $\boldsymbol{R}$ satisfies



where $\hat{A}_r(z)$ has its zeros on the unit circle

Reference input examples

a) Constant disturbance:

$$r(k+1) = r(k)$$

Then,

$$\boxed{\hat{A}_r(z) = z - 1}$$

b) Sinusoidal reference of ***<u>known</u>*** frequency:

$$r(k) = D\ \sin(\omega\, k + \phi)$$

Then,

$$\boxed{\hat{A}_r(z) = z^2 - 2\cos(\omega)\, z + 1}$$

Reference input examples

c) Periodic reference of ***known*** period N

$$r(k+N) = r(k)$$

Then,

$$\boxed{\widehat{A}_r(z) = z^N - 1}$$

In all of these three examples, the polynomial $\widehat{A}_r(z)$ has its zeros on the unit circle.

FSLQR with reference input

- Define the reference frequency weight

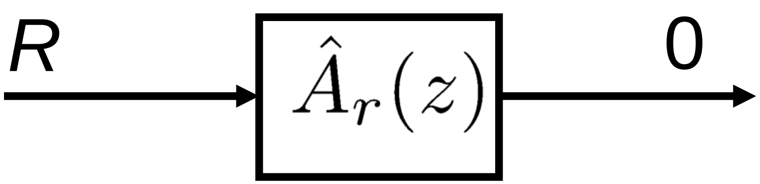
$$Q_R(e^{j\omega}) = Q_r^*(e^{j\omega})Q_r(e^{j\omega})$$

where

$$Q_r(z) = \frac{\widehat{B}_r(z)}{\widehat{A}_r(z)}$$

We can choose this

This is determined by the reference we are trying to follow



Frequency-Shaped Cost Function

$$J = \int_{-\pi}^{\pi} \Big\{ X^*(e^{j\omega}) Q(e^{j\omega}) X(e^{j\omega}) + U^*(e^{j\omega}) R(e^{j\omega}) U(e^{j\omega}) \Big\} d\omega$$

- with

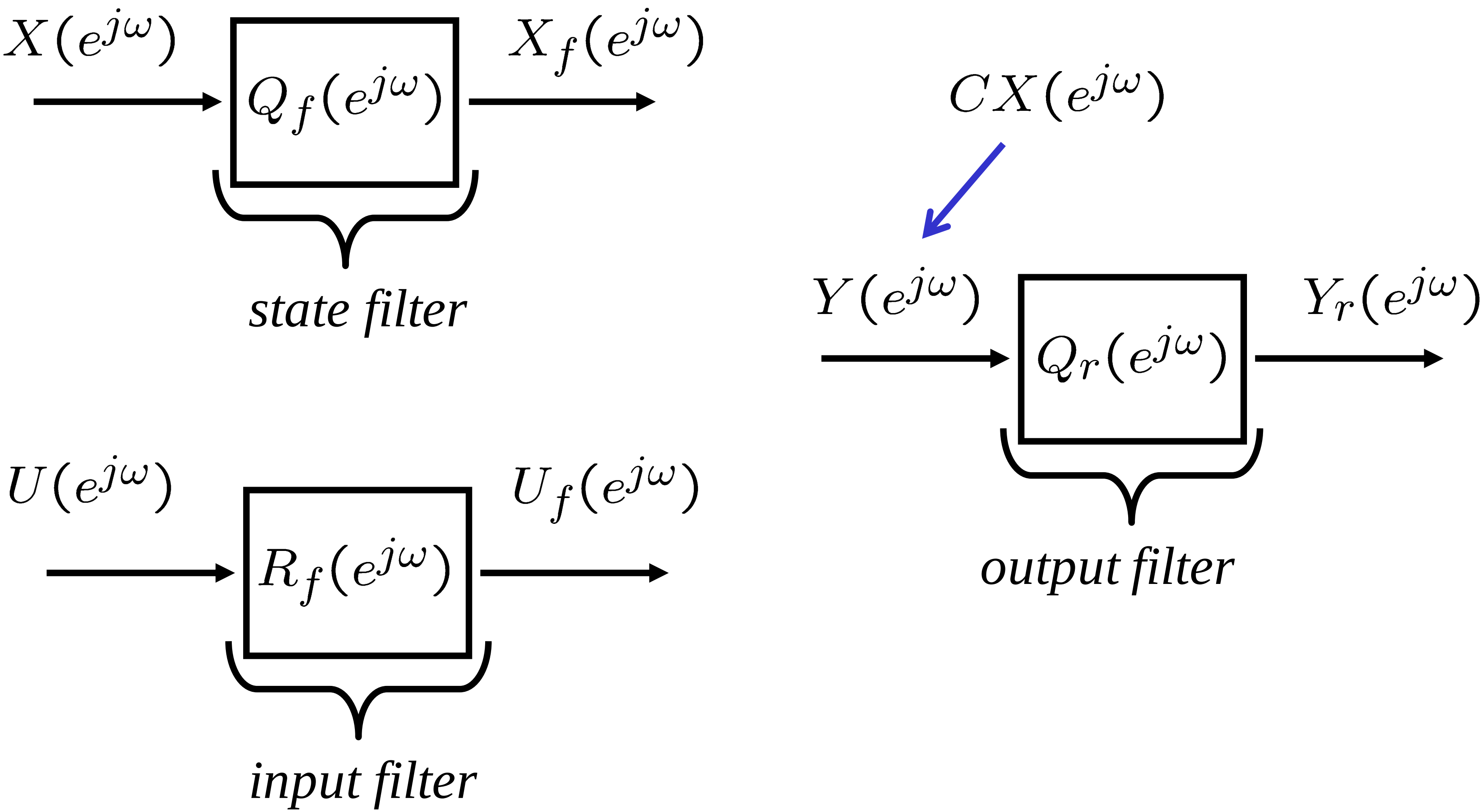
$$R(e^{j\omega}) = R_f^*(e^{j\omega}) R_f(e^{j\omega})$$

$$Q(e^{j\omega}) = \underbrace{C^T Q_r^*(e^{j\omega}) Q_r(e^{j\omega}) C}_{\text{used for achieving } \lim_{k\to\infty} e(k) = 0} + Q_f^*(e^{j\omega}) Q_f(e^{j\omega})$$

(we will show why later)

Frequency-Shaped Cost Function

Define the state, input, and output filters



Frequency-Shaped Cost Function

$$J = \int_{-\pi}^{\pi} \Big\{ X^*(e^{j\omega}) C^T Q_r^*(e^{j\omega}) Q_r(e^{j\omega}) C X(e^{j\omega})$$

$$+ X^*(e^{j\omega}) Q_f^*(e^{j\omega}) Q_f(e^{j\omega}) X(e^{j\omega})$$

$$+ U^*(e^{j\omega}) R_f^*(e^{j\omega}) R_f(e^{j\omega}) U(e^{j\omega}) \Big\} d\omega$$

can be written

$$J = \int_{-\pi}^{\pi} \Big\{ Y_r^*(e^{j\omega}) Y_r(e^{j\omega}) + X_f^*(e^{j\omega}) X_f(e^{j\omega}) + U_f^*(e^{j\omega}) U_f(e^{j\omega}) \Big\} d\omega$$

Realizing the filters using LTI's

Let

$$X(e^{j\omega}) \longrightarrow \boxed{Q_f(e^{j\omega})} \longrightarrow X_f(e^{j\omega})$$

be realized by

$$z_1(k+1) = A_1 z_1(k) + B_1 x(k)$$

$$x_f(k) = C_1 z_1(k) + D_1 x(k)$$

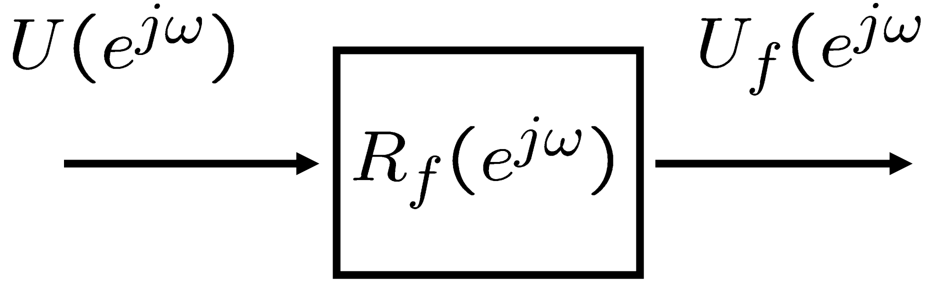
so that

$$Q_f(z) = C_1(zI - A_1)^{-1} B_1 + D_1$$

is causal or strictly causal.

Realizing the filters using LTI's

Let



be realized by

$$z_2(k+1) = A_2 z_2(k) + B_2 u(k)$$
$$u_f(k) = C_2 z_2(k) + D_2 u(k)$$

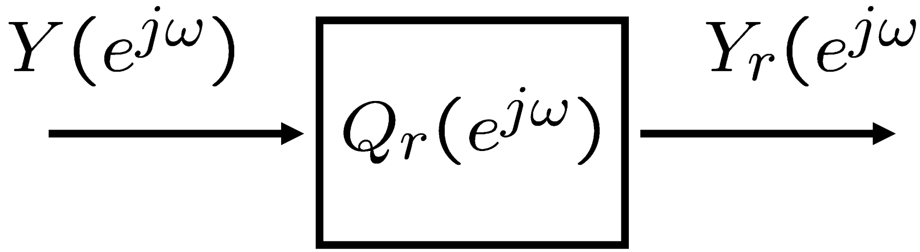
(with $D_2^T D_2 \succ 0$) so that

$$R_f(z) = C_2(zI - A_2)^{-1} B_2 + D_2$$

is causal (but not strictly causal)

Realizing the filters using LTI's

Let



be realized by

$$z_r(k+1) = A_r z_r(k) + B_r y(k)$$

$$y_r(k) = C_r z_r(k) + D_r y(k)$$

so that

$$Q_r(z) = C_r(zI - A_r)^{-1}B_r + D_r = \frac{\widehat{B}_r(z)}{\widehat{A}_r(z)}$$

is causal or strictly causal.

Cost Function Realization

Using Parseval's theorem,

$$J = \sum_{k=0}^{\infty} \left\{ y_r^T(k) y_r(k) + x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \right\}$$

where,

$$\begin{bmatrix} x(k+1) \\ z_r(k+1) \\ z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ B_r C & A_r & 0 & 0 \\ B_1 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} x(k) \\ z_r(k) \\ z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \\ B_2 \end{bmatrix} u(k)$$

$$\begin{bmatrix} y_r(k) \\ x_f(k) \\ u_f(k) \end{bmatrix} = \begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix} \begin{bmatrix} x(k) \\ z_r(k) \\ z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ D_2 \end{bmatrix} u(k)$$

Extended System Dynamics

$$\underbrace{\begin{bmatrix} x(k+1) \\ z_r(k+1) \\ z_1(k+1) \\ z_2(k+1) \end{bmatrix}}_{x_e(k+1)} = \underbrace{\begin{bmatrix} A & 0 & 0 & 0 \\ B_rC & A_r & 0 & 0 \\ B_1 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(k) \\ z_r(k) \\ z_1(k) \\ z_2(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \\ B_2 \end{bmatrix}}_{B_e} u(k)$$

$$x_e(k+1) = A_e x_e(k) + B_e u(k)$$

Extended System Cost

$$J = \sum_{k=0}^{\infty} \left\{ y_r^T(k) y_r(k) + x_f^T(k) x_f(k) + u_f^T(k) u_f(k) \right\}$$

Using

$$\begin{bmatrix} y_r(k) \\ x_f(k) \\ u_f(k) \end{bmatrix} = \underbrace{\begin{bmatrix} D_r C & C_r & 0 & 0 \\ D_1 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix}}_{C_e} \underbrace{\begin{bmatrix} x(k) \\ z_r(k) \\ z_1(k) \\ z_2(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ D_2 \end{bmatrix}}_{D_e} u(k)$$

the cost can be expressed

$$J = \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C_e^T \\ D_e^T \end{bmatrix} \begin{bmatrix} C_e & D_e \end{bmatrix} \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix} \right\}$$

FSLQR with reference input

Minimize

$$J = \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C_e^T \\ D_e^T \end{bmatrix} \begin{bmatrix} C_e & D_e \end{bmatrix} \begin{bmatrix} x_e(k) \\ u(k) \end{bmatrix} \right\}$$

Subject to

$$x_e(k+1) = A_e x_e(k) + B_e u(k)$$

This is a standard LQR problem!

Solution

The optimal control law is

$$u^o(k) = -K_e x_e(k)$$

$$K_e = [B_e^T P B_e + D_e^T D_e]^{-1} [B_e^T P A_e + D_e^T C_e]$$

where *P* is the solution of the DARE

$$\begin{aligned} P = {} & A_e^T P A_e + C_e^T C_e \\ & -[A_e^T P B_e + C_e^T D_e][B_e^T P B_e + D_e^T D_e]^{-1}[B_e^T P A_e + D_e^T C_e] \end{aligned}$$

for which $A_e - B_e K_e$ is Schur

Existence

The optimal control law exists if

- (A_e, B_e) stabilizable
- The state-space realization $C_e(zI-A_e)^{-1}B_e + D_e$ has no transmission zeros on the unit circle

Sufficient conditions for FSLQR

The optimal control law exists if the following hold:

1. (A,B) is stabilizable

2. Q_f and R_f are stable (i.e. A_1 and A_2 are Schur)

3. $\text{nullity}\begin{bmatrix} A_2 - \lambda I & B_2 \\ C_2 & D_2 \end{bmatrix} = 0$ whenever $|\lambda| = 1$

4. $\text{nullity}\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix} = 0$ whenever $\begin{cases} \det(A - \lambda I) = 0 \\ |\lambda| = 1 \end{cases}$

Sufficient conditions for FSLQR

The optimal control law exists if the following hold:

5. (A_r, B_r) is stabilizable

6. (C_r, A_r) has no unobservable modes on the unit circle

7. $\text{nullity}\left(\begin{bmatrix} A-\lambda I & B \\ C & 0 \end{bmatrix}^T\right) = 0$ whenever $\begin{cases} \det(A_r - \lambda I) = 0 \\ |\lambda| \geq 1 \end{cases}$

Remarks on existence conditions

- Conditions 1-4 are the same as for the FSLQR without a reference input
- Conditions 5-6 are met if the realization of Q_r is minimal
- Condition 7 is a <u>stronger</u> version of the condition that none of the unit circle or unstable eigenvalues of A_r are transmission zeros of $C(zI\text{-}A)^{-1}B$, the open-loop relationship between u and y
  - The condition here is not enough to guarantee FSLQR existence for reference tracking

Implementation

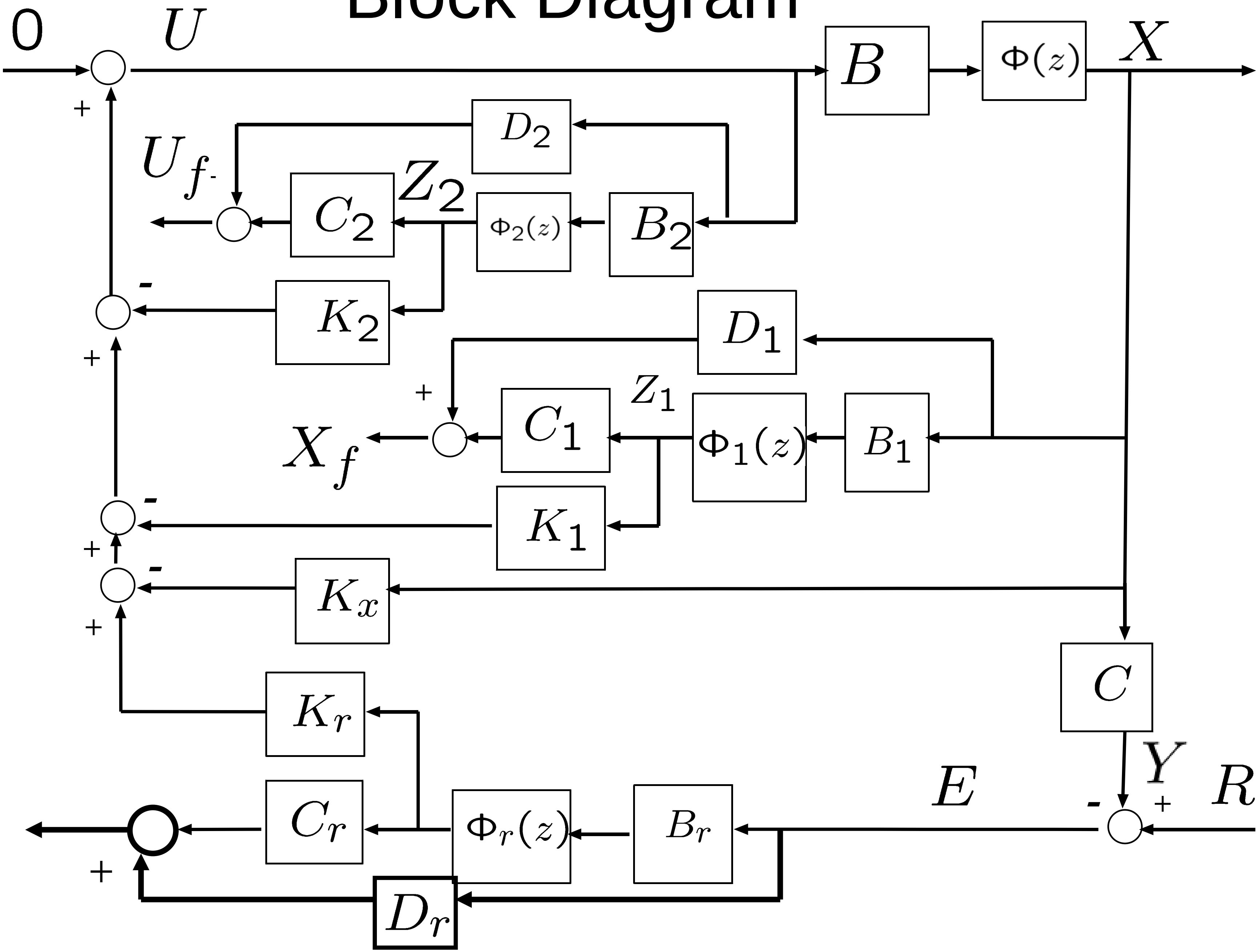
- Control

$$u(k) = -K_e x_e(k)$$

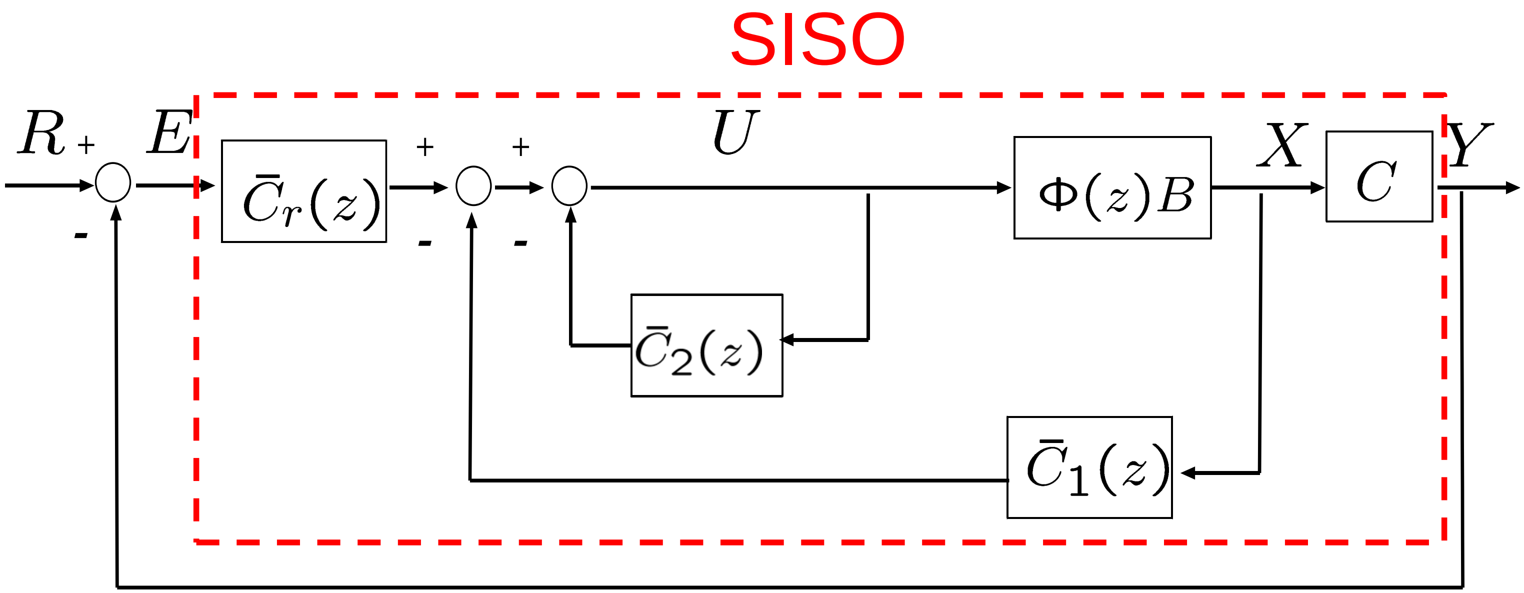
$$= -\begin{bmatrix} K_x & K_r & K_1 & K_2 \end{bmatrix} \begin{bmatrix} x(k) \\ z_r(k) \\ z_1(k) \\ z_2(k) \end{bmatrix}$$

$$= -K_x x(k) - K_r z_r(k) - K_1 z_1(k) - K_2 z_2(k)$$

Block Diagram



FSLQR with reference input – Block Diagram

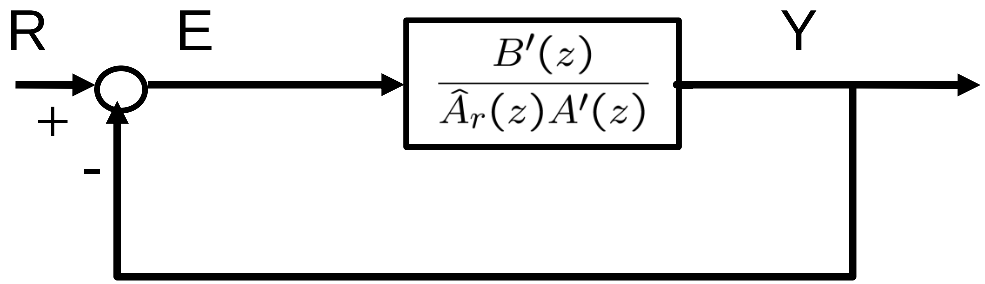


where

$$\bar{C}_1(z) = K_x + K_1\Phi_1(z)B_1 \qquad \bar{C}_2(z) = K_2\Phi_2(z)B_2$$

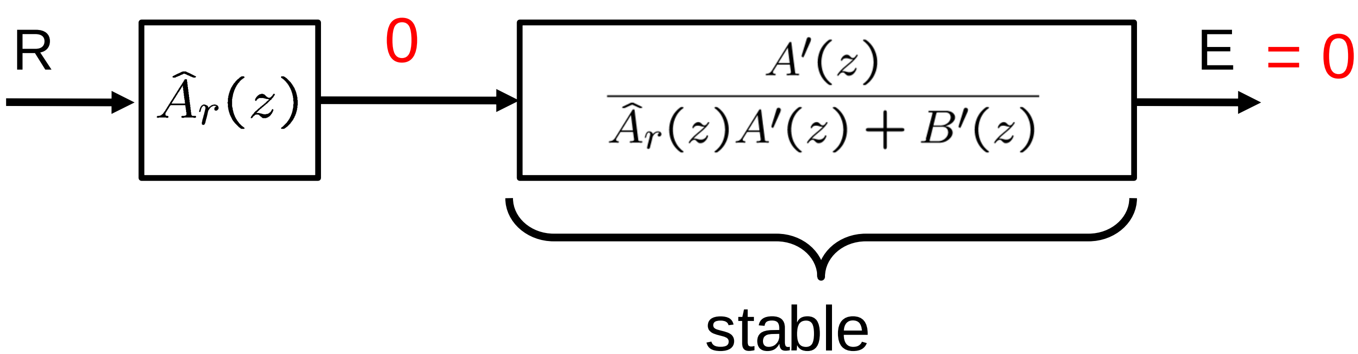
$$\bar{C}_r(z) = K_r\Phi_r(z)B_r = \frac{1}{\hat{A}_r(z)}\bar{B}_r(z)$$

FSLQR with reference input – Block Diagram



The closed-loop dynamics from *R* to *E* will be

$$G_{ER}(z) = \frac{1}{1 + \frac{B'(z)}{\widehat{A}_r(z)A'(z)}} = \frac{\widehat{A}_r(z)A'(z)}{\widehat{A}_r(z)A'(z) + B'(z)}$$



Course Outline

- Unit 0: Probability
- Unit 1: State-space control,estimation

Finished

- Unit 2: Input/output control
- Unit 3: Adaptive control

What we covered in Unit 1

Finite-horizon results

- Kalman filter
- Optimal LQR
- Optimal LQG
  - state feedback
  - output feedback

Infinite-horizon results

- Optimal LQR
- Kalman filter
- Optimal LQG
  - output feedback
- Frequency-shaped LQR

What we are skipping in Unit 1

- Continuous-time versions of:
  - Kalman filter
  - Optimal LQG
  - Frequency-shaped LQR
- Loop transfer recovery

Slides will be posted for reference

What we will cover in Unit 2

A collection of SISO input/output control design techniques

- Disturbance observer
- Pole placement, disturbance rejection, and tracking control
- Repetitive control and the internal model principle
- Minimum variance regulators