ME 233 Advanced Control II

Lecture 12
Stationary
Linear Quadratic Gaussian (LQG) Optimal Control

(ME233 Class Notes pp.LQG1-LQG7)
Outline

• Stationary LQG

• Relationship to $H_2$ optimal control
Stationary random inputs

Linear system contaminated by noise:

\[ Y = (zI - A)^{-1}CB + U + W + V \]

Assume that both

- \( w(k) \) and \( v(k) \) are WSS, zero-mean
Stationary LQG

We want to regulate the state

\[ x(k + 1) = A x(k) + B u(k) + B_w w(k) \]

\[ y(k) = C x(k) + v(k) \]

under

\[ E\{w(k)\} = 0 \]
\[ E\{v(k)\} = 0 \]
\[ E\{w(k + l)w^T(k)\} = W \delta(l) \]
\[ E\{v(k + l)v^T(k)\} = V \delta(l) \]
\[ E\{w(k + l)v^T(k)\} = 0 \]

WSS zero-mean white Gaussian Noise
Stationary LQG

\[ J = E \left\{ x^T(N) Q_f x(N) + \sum_{k=0}^{N-1} \left[ x^T(k) Q x(k) + u^T(k) R u(k) \right] \right\} \]

\[ Q = C_Q^T C_Q \]

Define the “incremental” cost

\[ J' = \frac{1}{N} J \]

The control that minimizes \( J \) also minimizes \( J' \)
Stationary LQG

“Incremental” cost:

\[
J' = E \left\{ \frac{1}{N} x^T(N) Q_f x(N) + \frac{1}{N} \sum_{k=0}^{N-1} \left[ x^T(k) C_Q^T C_Q x(k) + u^T(k) R u(k) \right] \right\}
\]

Under the stationarity assumptions:

\[
\lim_{N \to \infty} J' = J_s
\]

\[
J_s = E \left\{ x^T(k) C_Q^T C_Q x(k) + u^T(k) R u(k) \right\}
\]
Stationary LQG

Obtain the optimal control that minimizes:

\[ J_s = E\{x^T(k)C_Q^T C_Q x(k) + u^T(k)R u(k)\} \]

under

\[ x(k+1) = A x(k) + B u(k) + B_w w(k) \]
\[ y(k) = C x(k) + v(k) \]

- \( w(k) \) and \( v(k) \) are WSS
Optimal stationary LQG

Theorem:

a) The optimal control is given by

\[ u^o(k) = -K \hat{x}(k) \]

\[ K = \left( B^T P B + R \right)^{-1} B^T P A \]

\[ P = A^T P A + Q - A^T P B \left( B^T P B + R \right)^{-1} B^T P A \]

Such that \( A - B K \) is Schur

*Standard deterministic infinite-horizon LQR solution!*
Optimal stationary LQG

Theorem (cont’d): \[ u^o(k) = -K \hat{x}(k) \]

A-posteriori state observer structure:

\[ \hat{x}(k) = \hat{x}^o(k) + F\tilde{y}(k) \]
\[ \hat{x}^o(k + 1) = A\hat{x}(k) + Bu(k) \]
\[ \tilde{y}^o(k) = y(k) - C\hat{x}^o(k) \]

\[ F = MC^T \left[ CMC^T + V \right]^{-1} \]
\[ M = AMAT^T + B_wWB_w^T - AMCT \left[ CMC^T + V \right]^{-1} CMAT \]

Such that \( A - (AF)C \) is Schur

\[ L = AMCT \left[ CMC^T + V \right]^{-1} \]
State space form of LQG controller

\[
\hat{x}^o(k + 1) = [A - LC] \hat{x}^o(k) + Bu(k) + Ly(k)
\]
\[
\hat{x}(k) = [I - FC] \hat{x}^o(k) + Fy(k)
\]
\[
u^o(k) = -K \hat{x}(k)
\]

Eliminating \( \hat{x}(k) \) from the expression for \( u^o(k) \) yields

\[
u^o(k) = -K[I - FC] \hat{x}^o(k) - KFy(k)
\]

Plugging this expression for \( u^o(k) \) into the expression for \( \hat{x}^o(k + 1) \) yields the state space model on the next slide.
State space form of LQG controller

\[
\hat{x}^o(k+1) = [A - LC - BK + BKFC]\hat{x}^o(k) + [L - BKF]y(k)
\]
\[
u^o(k) = [-K + KFC]\hat{x}^o(k) - KFy(k)
\]

\textit{K} is the standard deterministic LQR gain
\textit{F} and \textit{L} are the standard Kalman filter gains

The closed-loop poles are the eigenvalues of \(A - BK\) and the eigenvalues of \(A - LC\)
Optimal stationary LQG

Theorem (cont’d):

b) The optimal cost is

\[ J_s^o = \text{trace} \left\{ P \left[ BKZA^T + B_wW B_w^T \right] \right\} \]

\[ Z = E\{\ddot{x}(k)\ddot{x}^T(k)\} \]

(see the derivation of this result at the end)
Conditions for existence

• Existence of infinite-horizon LQR solution
  – \((A, B)\) stabilizable
  – \((C_Q, A)\) has no unobservable modes on the unit circle

• Existence of stationary KF solution
  – \((C, A)\) detectable
  – \((A, B W^{1/2})\) has no uncontrollable modes on the unit circle
**$H_2$ norm**

- Let $G(z)$ be a **stable** discrete-time transfer function

- The $H_2$ norm of $G(z)$ is defined by

$$\|G(z)\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}[G(e^{j\omega})G^*(e^{j\omega})]d\omega$$

  $$= \text{trace}\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})G^*(e^{j\omega})d\omega\right]$$
\[ H_2 \text{ norm} \]

\[
\|G(z)\|_2^2 = \text{trace} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})G^*(e^{j\omega}) d\omega \right]
\]

Suppose \( U(k) \) is WSS and zero-mean,

\[
U(z) \rightarrow G(z) \rightarrow Y(z)
\]

\( \Lambda_{UU}(j) = I \delta(j) \)

Then \( \Phi_{UU}(\omega) = I \)

\[
\Rightarrow \Phi_{YY}(\omega) = G(e^{j\omega})\Phi_{UU}(\omega)G^*(e^{j\omega})
\]

\[
= G(e^{j\omega})G^*(e^{j\omega})
\]
The $H_2$ norm of $G(z)$ can be written in this form:

$$\|G(z)\|_2^2 = \text{trace} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{yy}(\omega) d\omega \right] \Lambda_{yy}(0)$$

$$= \text{trace}[E\{Y(k)Y^T(k)\}]$$

$$= E\{Y^T(k)Y(k)\}$$

LQG cost function can be written in this form:

$$\Lambda_{uu}(j) = I \delta(j)$$
Plant dynamics

\[ x(k + 1) = A x(k) + B u(k) + B_w w(k) \]
\[ y(k) = C x(k) + v(k) \]

define \[ \bar{w}(k) = \begin{bmatrix} W^{-1/2} & w(k) \\ V^{-1/2} & v(k) \end{bmatrix} \]

\[ x(k + 1) = A x(k) + B u(k) + \begin{bmatrix} B_w W^{1/2} & 0 \end{bmatrix} \bar{w}(k) \]
\[ y(k) = C x(k) + \begin{bmatrix} 0 & V^{1/2} \end{bmatrix} \bar{w}(k) \]
Noise covariance

\[ \bar{w}(k) = \begin{bmatrix} W^{-1/2} & w(k) \\ V^{-1/2} & v(k) \end{bmatrix} = \begin{bmatrix} W^{-1/2} & 0 \\ 0 & V^{-1/2} \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \]

\[ \Lambda_{\bar{w}\bar{w}}(j) = \begin{bmatrix} W^{-1/2} & 0 \\ 0 & V^{-1/2} \end{bmatrix} E \left\{ \begin{bmatrix} w(k+j) \\ v(k+j) \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}^T \right\} \begin{bmatrix} W^{-1/2} & 0 \\ 0 & V^{-1/2} \end{bmatrix} \]

\[ \Lambda_{\bar{w}\bar{w}}(j) = I \delta(j) \]
Stationary LQG cost function

\[ J_s = E\{x^T(k)C_Q^T C_Q x(k) + u^T(k)Ru(k)\} \]

factor as \( D^T D \)

define \( p(k) = \begin{bmatrix} C_Q x(k) \\ D u(k) \end{bmatrix} \)

\[ p^T(k)p(k) = x^T(k)C_Q^T C_Q x(k) + u^T(k)Ru(k) \]

\[ J_s = E\{p^T(k)p(k)\} \]
Plant dynamics and LQG cost

\[ x(k + 1) = A x(k) + B u(k) + \begin{bmatrix} B_w W^{1/2} & 0 \end{bmatrix} \bar{w}(k) \]

\[ p(k) = \begin{bmatrix} C_Q \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ D \end{bmatrix} u(k) \]

\[ y(k) = C x(k) + \begin{bmatrix} 0 & V^{1/2} \end{bmatrix} \bar{w}(k) \]
**$H_2$ optimal control problem**

- For any given stabilizing LTI controller, the squared $H_2$ norm of the closed-loop system is $E\{p^T(k)p(k)\}$

  *This is equal to the stationary LQG cost!*

Minimizing the closed-loop $H_2$ norm is equivalent to minimizing the stationary LQG cost

\[ \Lambda_{\bar{w}\bar{w}}(j) = I \delta(j) \]
One way to choose an LQG cost function

\[ p(k) = [\alpha_1 p_1(k) \quad \alpha_2 p_2(k) \quad \cdots \quad \alpha_q p_q(k)]^T \]

Each \( p_i(k) \) is a signal you would like to keep “small” in the closed-loop system

e.g. position error, control effort, actuator displacement

Always include control effort!
One way to choose an LQG cost function

\[ p(k) = [\alpha_1 p_1(k), \alpha_2 p_2(k), \ldots, \alpha_q p_q(k)]^T \]

\[ p^T(k)p(k) = \sum_{i=1}^{q} \alpha_i^2 p_i^2(k) \]

\[ J_s = \sum_{i=1}^{q} \alpha_i^2 E\{p_i^2(k)\} \]

For any chosen nonzero values of \( \alpha_1, \ldots, \alpha_q \), you can perform an optimal control design and then find the values of \( E\{p_1^2(k)\}, \ldots, E\{p_q^2(k)\} \)

Choose nonzero values of \( \alpha_1, \ldots, \alpha_q \) so that the values of \( E\{p_1^2(k)\}, \ldots, E\{p_q^2(k)\} \) are reasonable

This requires iteration
Additional material
(you are not responsible for this)

• Derivation of optimal stationary LQG cost
Stationary LQG

Optimal cost (derivation)

The incremental optimal cost is

\[
J^0_s = \lim_{N \to \infty} \frac{1}{N} \left\{ \hat{J}^0 + \sum_{j=0}^{N-1} \text{trace}[QZ(j)] + \text{trace}[Q_fZ(N)] \right\}
\]

\[
\hat{J}^0 = x_o^T P(0) x_o + \text{trace} \left[ P(0) \bar{X}_o \right] + \hat{b}(0)
\]

\[
\hat{b}(k - 1) = \hat{b}(k) + \text{trace} \left[ F^T(k) P(k) F(k) [CM(k) C^T + V] \right]
\]

Thus

\[
J^0_s = \text{Tr} \left\{ \left[ QZ + F^T PF \left[ CM C + V \right] \right] \right\}
\]
Stationary LQG

Optimal cost (derivation)

\[ J_s^O = \text{Tr} \left\{ [QZ + F^T PF][CMC + V] \right\} \]

Note:

\[ A^T PA - P = -Q + A^T PB \left[ B^T PB + R \right]^{-1} B^T PA \]

\[ F = MCT \left[ CMC^T + V \right]^{-1} \]

\[ Z = M - MCT \left[ CMC^T + V \right]^{-1} CM \quad \text{(least squares)} \]

\[ M = AZA^T + B_wWB_w^T \]
Stationary LQG

Optimal cost (derivation)

\[ J_s^o = \text{Tr} \left\{ [QZ + F^T PF][CMC + V] \right\} \]

last term:

\[ \text{Tr} \left\{ F^T PF [CMC + V] \right\} = \]
\[ = \text{Tr} \{ F^T PMC^T \} = \text{Tr} \{ PMC^T F^T \} \]
\[ = \text{Tr} \{ PMC^T [CMC^T + V]^{-1} CM \} \]
\[ = \text{Tr} \{ P(M - Z) \} \]

first term:

\[ \text{Tr}\{QZ\} = \]
\[ = \text{Tr}\{ [P - A^T PA + A^T PB[B^T PB + R]^{-1} B^T PA] Z \} \]
\[ = \text{Tr}\{ PZ + [-PA + PBK] ZA^T \} \]
Stationary LQG

Optimal cost (derivation)

\[
J_s^O = \text{Tr} \left\{ QZ + F^T PF[CMC + V] \right\}
\]

\[
J_s^O = \text{Tr}\{PZ + [-PA + PBK]ZA^T + P(M - Z)\}
\]

\[
J_s^O = \text{Tr}\{[-PA + PBK]ZA^T - P[AZA^T + B_wWB_w^T]\}
\]

\[
= \text{Tr}\{PB_KZA^T + PB_wWB_w^T\}
\]