ME 233 Advanced Control II

Lecture 12
Stationary
Linear Quadratic Gaussian (LQG)
Optimal Control

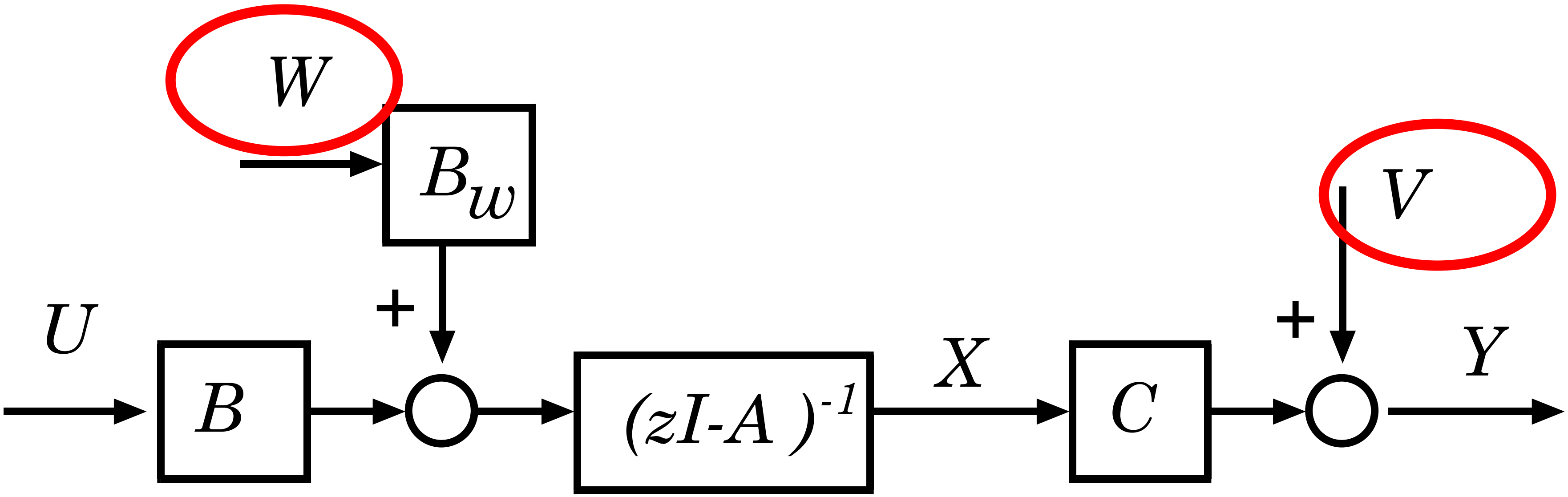
(ME233 Class Notes pp.LQG1-LQG7)

Outline

- Stationary LQG
- Relationship to H_2 optimal control

Stationary random inputs

Linear system contaminated by noise:



Assume that both

- $\boldsymbol{w(k)}$ and $\boldsymbol{v(k)}$ are WSS, zero-mean

Stationary LQG

We want to regulate the state

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\
y(k) &= C\,x(k) + v(k)
\end{aligned}
$$

under

$$
\left.
\begin{aligned}
E\{w(k)\} &= 0 \\
E\{v(k)\} &= 0 \\
E\{w(k+l)w^T(k)\} &= W\,\delta(l) \\
E\{v(k+l)v^T(k)\} &= V\,\delta(l) \\
E\{w(k+l)v^T(k)\} &= 0
\end{aligned}
\right\}
$$

WSS zero-mean white Gaussian Noise

Stationary LQG

$$J = E\left\{x^T(N)\,Q_f\,x(N) + \sum_{k=0}^{N-1}\left[x^T(k)\,Q\,x(k) + u^T(k)\,R\,u(k)\right]\right\}$$

$$Q = C_Q^T C_Q$$

Define the "incremental" cost

$$J^{'} = \frac{1}{N}\,J$$

The control that minimizes J also minimizes $J^{'}$

Stationary LQG

"Incremental" cost:

$$J' = E\left\{\frac{1}{N}\, x^T(N)\, Q_f\, x(N) + \frac{1}{N}\sum_{k=0}^{N-1}\left[x^T(k)\, C_Q^T C_Q\, x(k) + u^T(k)\, R\, u(k)\right]\right\}$$

Under the stationarity assumptions:

$$\lim_{N\to\infty} J' = J_s$$

$$J_s = E\{x^T(k) C_Q^T C_Q x(k) + u^T(k) R u(k)\}$$

Stationary LQG

Obtain the optimal control that minimizes:

$$J_s = E\{x^T(k) C_Q^T C_Q x(k) + u^T(k) R u(k)\}$$

under

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

- $\boldsymbol{w(k)}$ and $\boldsymbol{v(k)}$ are WSS

Optimal stationary LQG

Theorem:

a) The optimal control is given by

$$u^o(k) = -\boxed{K}\ \widehat{x}(k)$$

$$K = \left[B^T P B + R\right]^{-1} B^T P A$$

$$P = A^T P A + Q - A^T P B [B^T P B + R]^{-1} B^T P A$$

Such that $A - BK$ is Schur

Standard deterministic infinite-horizon LQR solution!

Optimal stationary LQG

Theorem (cont'd): $$u^o(k) = -K \boxed{\hat{x}(k)}$$

A-posteriori state observer structure:

$$\boxed{\begin{aligned} \hat{x}(k) &= \hat{x}^o(k) + F\tilde{y}(k) \\ \hat{x}^o(k+1) &= A\hat{x}(k) + Bu(k) \\ \tilde{y}^o(k) &= y(k) - C\hat{x}^o(k) \end{aligned}}$$

$$F = MC^T\left[CMC^T + V\right]^{-1}$$

$$M = AMA^T + B_w W B_w^T - AMC^T\left[CMC^T + V\right]^{-1} CMA^T$$

Such that $A - \underbrace{(AF)}_{L = AMC^T\left[CMC^T + V\right]^{-1}}C$ is Schur

State space form of LQG controller

$$\left.\begin{aligned} \hat{x}^o(k+1) &= [A - LC]\hat{x}^o(k) + Bu(k) + Ly(k) \\ \hat{x}(k) &= [I - FC]\hat{x}^o(k) + Fy(k) \end{aligned}\right\} \text{Kalman filter}$$

$$\left.\begin{aligned} u^o(k) &= -K\hat{x}(k) \end{aligned}\right\} \text{LQR}$$

Eliminating $\hat{x}(k)$ from the expression for $u^o(k)$ yields

$$u^o(k) = -K[I - FC]\hat{x}^o(k) - KFy(k)$$

Plugging this expression for $u^o(k)$ into the expression for $\hat{x}^o(k+1)$ yields the state space model on the next slide

State space form of LQG controller

$$\boxed{\begin{aligned} \hat{x}^o(k+1) &= [A - LC - BK + BKFC]\hat{x}^o(k) + [L - BKF]y(k) \\ u^o(k) &= [-K + KFC]\hat{x}^o(k) - KFy(k) \end{aligned}}$$

K is the standard deterministic LQR gain

F and *L* are the standard Kalman filter gains

The closed-loop poles are the eigenvalues of *A* – *BK* and the eigenvalues of *A* – *LC*

Optimal stationary LQG

Theorem (cont'd):

b) The optimal cost is

$$J_s^o = \mathsf{trace}\left\{P\left[BKZA^T + B_w W B_w^T\right]\right\}$$

$$Z = E\{\tilde{x}(k)\tilde{x}^T(k)\}$$

(see the derivation of this result at the end)

Conditions for existence

- Existence of infinite-horizon LQR solution
  - (A,B) stabilizable
  - (C_Q ,A) has no unobservable modes on the unit circle
- Existence of stationary KF solution
  - (C,A) detectable
  - $(A, B_W W^{1/2})$ has no uncontrollable modes on the unit circle

H_2 norm

- Let $G(z)$ be a stable discrete-time transfer function

- The H_2 norm of $G(z)$ is defined by

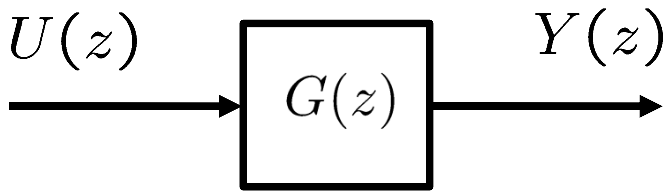
$$\|G(z)\|_2^2 = \overbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}}^{\text{Average over frequency}} \overbrace{\text{trace}[G(e^{j\omega})G^*(e^{j\omega})]}^{\text{Squared Frobenius norm of } G(e^{j\omega})} d\omega$$

$$= \text{trace}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})G^*(e^{j\omega})d\omega\right]$$

H_2 norm

$$\|G(z)\|_2^2 = \text{trace}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{G(e^{j\omega})G^*(e^{j\omega})}\,d\omega\right]$$

Suppose *U(k)* is WSS and zero-mean,



$$\Lambda_{UU}(j) = I\,\delta(j)$$

Then $\Phi_{UU}(\omega) = I$

$$\Rightarrow \quad \Phi_{YY}(\omega) = G(e^{j\omega})\Phi_{UU}(\omega)G^*(e^{j\omega})$$

$$= \underline{G(e^{j\omega})G^*(e^{j\omega})}$$

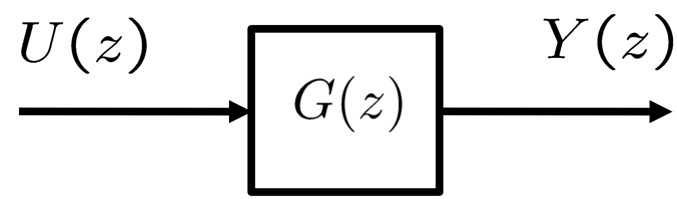
H_2 norm

$$\|G(z)\|_2^2 = \text{trace}\underbrace{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{YY}(\omega)d\omega\right]}_{\Lambda_{YY}(0)}$$

$$= \text{trace}[E\{Y(k)Y^T(k)\}]$$

$$= E\{Y^T(k)Y(k)\}]$$

LQG cost function can be written in this form



$$\Lambda_{UU}(j) = I\,\delta(j)$$

Plant dynamics

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k)$$
$$y(k) = C\,x(k) + v(k)$$

define $\bar{w}(k) = \begin{bmatrix} W^{-1/2}\ w(k) \\ V^{-1/2}\ v(k) \end{bmatrix}$

$$\Rightarrow \quad x(k+1) = A\,x(k) + B\,u(k) + \begin{bmatrix} B_w W^{1/2} & 0 \end{bmatrix} \bar{w}(k)$$
$$y(k) = C\,x(k) + \begin{bmatrix} 0 & V^{1/2} \end{bmatrix} \bar{w}(k)$$

Noise covariance

$$\bar{w}(k) = \begin{bmatrix} W^{-1/2}\, w(k) \\ V^{-1/2}\, v(k) \end{bmatrix} = \begin{bmatrix} W^{-1/2} & 0 \\ 0 & V^{-1/2} \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}$$

$$\Rightarrow \quad \Lambda_{\bar{w}\bar{w}}(j) = \begin{bmatrix} W^{-1/2} & 0 \\ 0 & V^{-1/2} \end{bmatrix} \underbrace{E\left\{ \begin{bmatrix} w(k+j) \\ v(k+j) \end{bmatrix} \begin{bmatrix} w(k) \\ v(k) \end{bmatrix}^T \right\}}_{\begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(j)} \begin{bmatrix} W^{-1/2} & 0 \\ 0 & V^{-1/2} \end{bmatrix}$$

$$\Rightarrow \quad \Lambda_{\bar{w}\bar{w}}(j) = I\, \delta(j)$$

Stationary LQG cost function

$$J_s = E\{x^T(k)C_Q^T C_Q x(k) + u^T(k)Ru(k)\}$$

factor as $D^T D$

define $$p(k) = \begin{bmatrix} C_Q\, x(k) \\ D\, u(k) \end{bmatrix}$$

$$\Rightarrow \quad p^T(k)p(k) = x^T(k)C_Q^T C_Q x(k) + u^T(k)Ru(k)$$

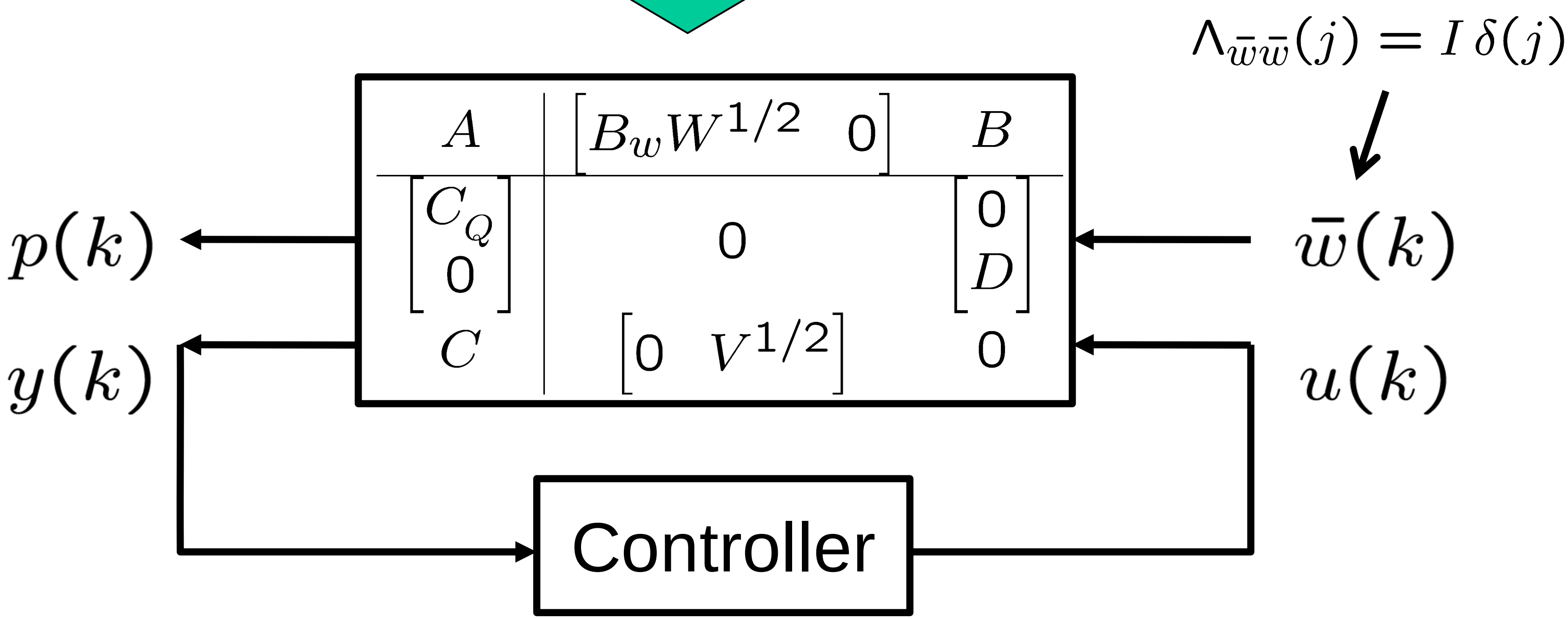
$$\Rightarrow \quad J_s = E\{p^T(k)p(k)\}$$

Plant dynamics and LQG cost

$$x(k+1) = A\,x(k) + B\,u(k) + \begin{bmatrix} B_w W^{1/2} & 0 \end{bmatrix} \bar{w}(k)$$

$$p(k) = \begin{bmatrix} C_Q \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ D \end{bmatrix} u(k)$$

$$y(k) = C\,x(k) + \begin{bmatrix} 0 & V^{1/2} \end{bmatrix} \bar{w}(k)$$



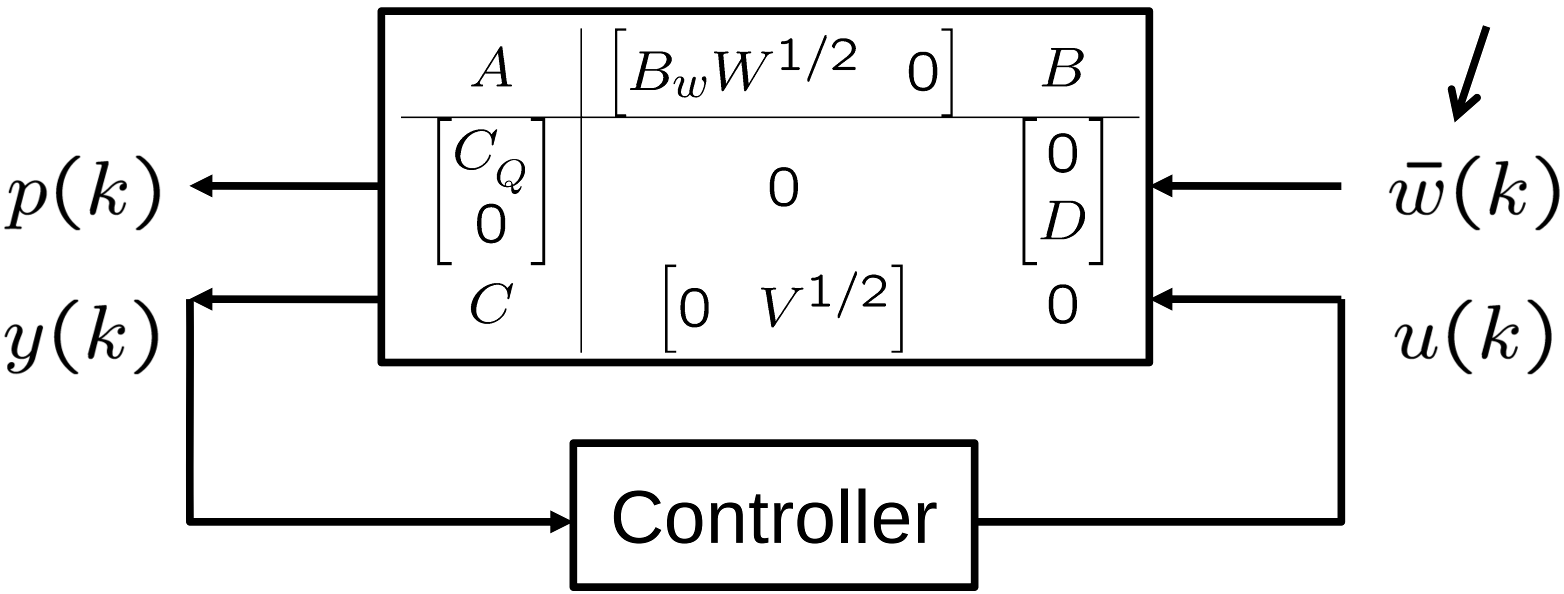
H_2 optimal control problem

- For any given <u>stabilizing</u> LTI controller, the squared H_2 norm of the closed-loop system is $E\{p^T(k)p(k)\}$

This is equal to the stationary LQG cost!

$\boxed{\text{Minimizing the closed-loop } H_2 \text{ norm is equivalent to minimizing the stationary LQG cost}}$

$$\Lambda_{\bar{w}\bar{w}}(j) = I\,\delta(j)$$



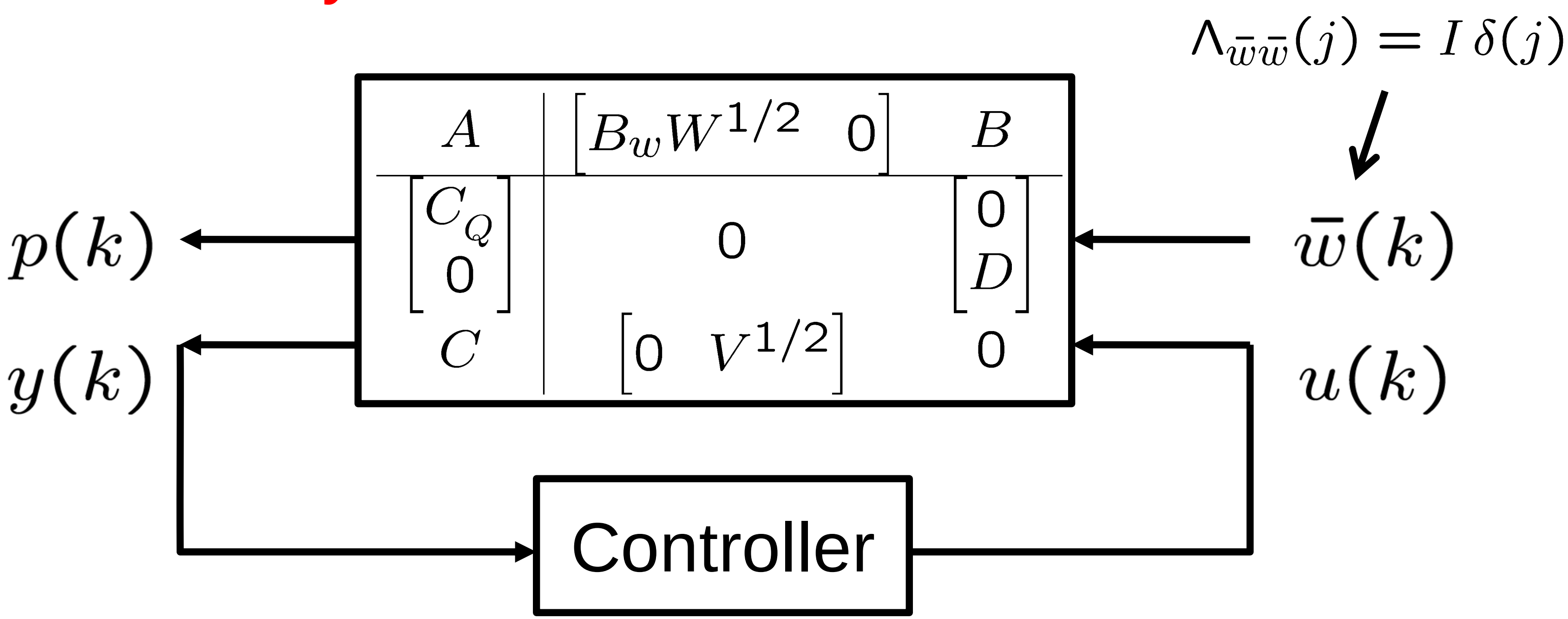
One way to choose an LQG cost function

$$p(k) = [\alpha_1 p_1(k) \;\; \alpha_2 p_2(k) \;\; \cdots \;\; \alpha_q p_q(k)]^T$$

Each $p_i(k)$ is a signal you would like to keep "small" in the closed-loop system

e.g. position error, control effort, actuator displacement

Always include control effort!



One way to choose an LQG cost function

$$p(k) = [\alpha_1 p_1(k) \;\; \alpha_2 p_2(k) \;\; \cdots \;\; \alpha_q p_q(k)]^T$$

$$\Rightarrow \quad p^T(k)p(k) = \sum_{i=1}^{q} \alpha_i^2 p_i^2(k)$$

$$\Rightarrow \quad J_s = \sum_{i=1}^{q} \alpha_i^2 E\{p_i^2(k)\}$$

For any chosen nonzero values of $\alpha_1, \ldots, \alpha_q$, you can perform an optimal control design and then find the values of $E\{p_1^2(k)\}, \ldots, E\{p_q^2(k)\}$

Choose nonzero values of $\alpha_1, \ldots, \alpha_q$ so that the values of $E\{p_1^2(k)\}, \ldots, E\{p_q^2(k)\}$ are reasonable

This requires iteration

Additional material (you are not responsible for this)

- Derivation of optimal stationary LQG cost

Stationary LQG

Optimal cost (derivation)

The incremental optimal cost is

$$J_s^o = \lim_{N\to\infty} \frac{1}{N}\left\{ \hat{J}^o + \sum_{j=0}^{N-1} \mathrm{trace}[QZ(j)] + \mathrm{trace}[Q_f Z(N)] \right\}$$

$$\hat{J}^o = x_o^T P(0) x_o + \mathrm{trace}\left[P(0)\bar{X}_o\right] + \hat{b}(0)$$

$$\hat{b}(k-1) = \hat{b}(k) + \mathrm{trace}\left[F^T(k)P(k)F(k)[CM(k)C^T + V]\right]$$

Thus

$$J_s^o = \mathrm{Tr}\left\{\left[QZ + F^T PF[CMC + V]\right]\right\}$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \mathrm{Tr}\left\{\left[QZ + F^T PF[CMC + V]\right]\right\}$$

Note:

$$A^T PA - P \;=\; -Q + A^T PB\left[B^T PB + R\right]^{-1} B^T PA$$

$$F \;=\; MC^T\left[C\,MC^T + V\right]^{-1}$$

$$Z = M - MC^T\left[CMC^T + V\right]^{-1} CM$$ (least squares)

$$M = AZA^T + B_w W B_w^T$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \text{Tr}\left\{\left[QZ + F^T PF[CMC + V]\right]\right\}$$

last term:

$$\text{Tr}\left\{F^T PF[CMC + V]\right\} =$$

$$= \text{Tr}\{F^T PMC^T\} = \text{Tr}\{PMC^T F^T\}$$

$$= \text{Tr}\{PMC^T[CMC^T + V]^{-1}CM\}$$

$$= \text{Tr}\{P(M - Z)\}$$

first term:

$$\text{Tr}\{QZ\} =$$

$$= \text{Tr}\{\left[P - A^T PA + A^T PB[B^T PB + R]^{-1} B^T PA\right] Z\}$$

$$= \text{Tr}\{PZ + [-PA + PBK]\, ZA^T\}$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \mathsf{Tr}\left\{QZ + F^T PF[CMC + V]\right\}$$

$$J_s^o = \mathsf{Tr}\{PZ + [-PA + PBK]\,ZA^T + P(M - Z)\}$$

$$\begin{aligned} J_s^o &= \mathsf{Tr}\{[-PA + PBK]ZA^T - P[AZA^T + B_w W B_w^T]\} \\ &= \mathsf{Tr}\{PBKZA^T + PB_w W B_w^T\} \end{aligned}$$