ME 233 Advanced Control II

Lecture 11
Kalman Filters Stationary Properties
and
LQR-KF Duality

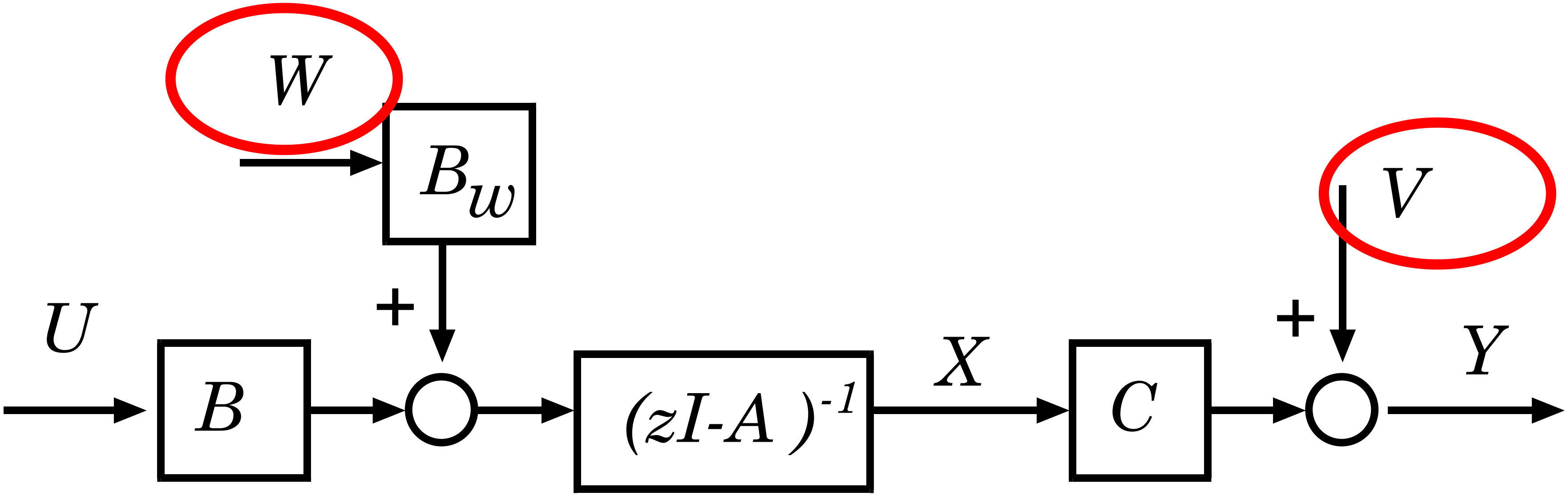
(ME233 Class Notes pp.KF1-KF6)

Summary

- Stationary Kalman filters (KF):
  - KF algebraic Riccati equation
  - Convergence properties
- Kalman filter / LQR duality
- KF return difference equality
  - Reciprocal root locus
  - Guaranteed robustness margins

Stochastic State Estimation

Linear system contaminated by noise:



Two random disturbances:

- Input noise $\boldsymbol{w(k)}$ - contaminates the state $\boldsymbol{x(k)}$
- Measurement noise $\boldsymbol{v(k)}$ - contaminates the output $\boldsymbol{y(k)}$

Stochastic state model

State estimation of LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) + B_w\,w(k)$$

$$y(k) = C\,x(k) + v(k)$$

Where:

- $u(k)$ **known control input**
- $w(k)$ Gaussian, uncorrelated, zero mean, input noise
- $v(k)$ Gaussian, uncorrelated, zero mean, meas. noise
- $x(0)$ Gaussian

Assumptions (review)

- Initial conditions:

$$E\{x(0)\} = x_o \qquad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o$$

- Noise properties:

$$\left.\begin{aligned} &E\{w(k)\} = 0 \\ &E\{v(k)\} = 0 \\ &E\{w(k+l)w^T(k)\} = W(k)\,\delta(l) \\ &E\{v(k+l)v^T(k)\} = V(k)\,\delta(l) \\ &E\{w(k+l)v^T(k)\} = 0 \end{aligned}\right\}$$

Zero-mean Gaussian uncorrelated noises

$$E\{\tilde{x}^o(0)w^T(k)\} = 0 \qquad E\{\tilde{x}^o(0)v^T(k)\} = 0$$

Kalman Filter Solution V-1 (review)

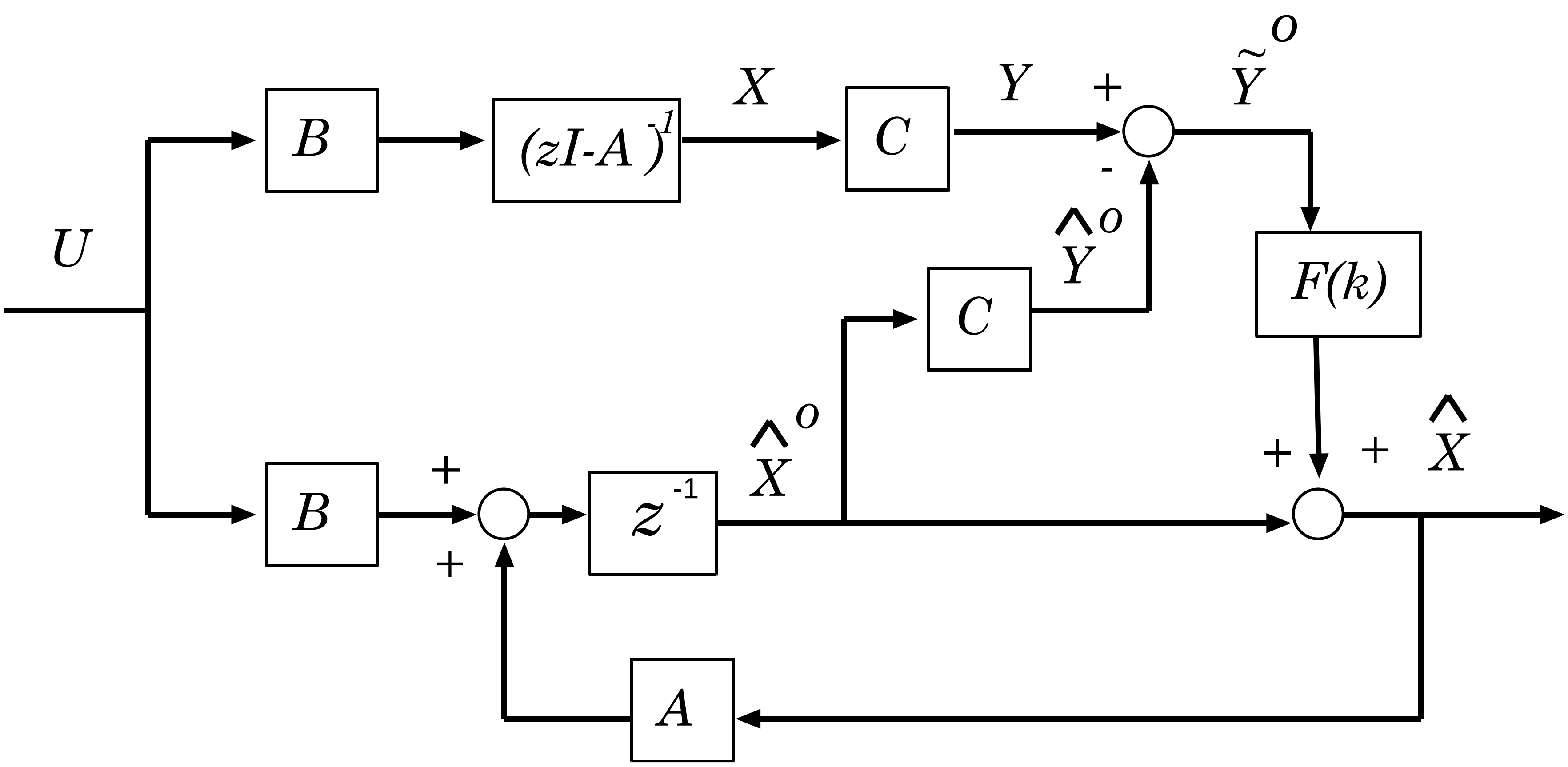
A-posteriori state observer structure:

$$\begin{aligned}
\widehat{x}(k) &= \widehat{x}^{o}(k) + F(k)\,\widetilde{y}^{o}(k) \\
\widehat{x}^{o}(k+1) &= A\,\widehat{x}(k) + B\,u(k) \\
\widetilde{y}^{o}(k) &= y(k) - C\,\widehat{x}^{o}(k)
\end{aligned}$$

$$\begin{aligned}
F(k) &= M(k)C^{T}\left[C\,M(k)C^{T} + V(k)\right]^{-1} \\
M(k+1) &= AM(k)A^{T} + B_{w}W(k)B_{w}^{T} \\
&\quad - AM(k)C^{T}\left[CM(k)C^{T} + V(k)\right]^{-1} CM(k)A^{T}
\end{aligned}$$

Kalman Filter Solution V-1 (review)

- A-posteriori estimator as output



Kalman Filter Solution V-2 (review)

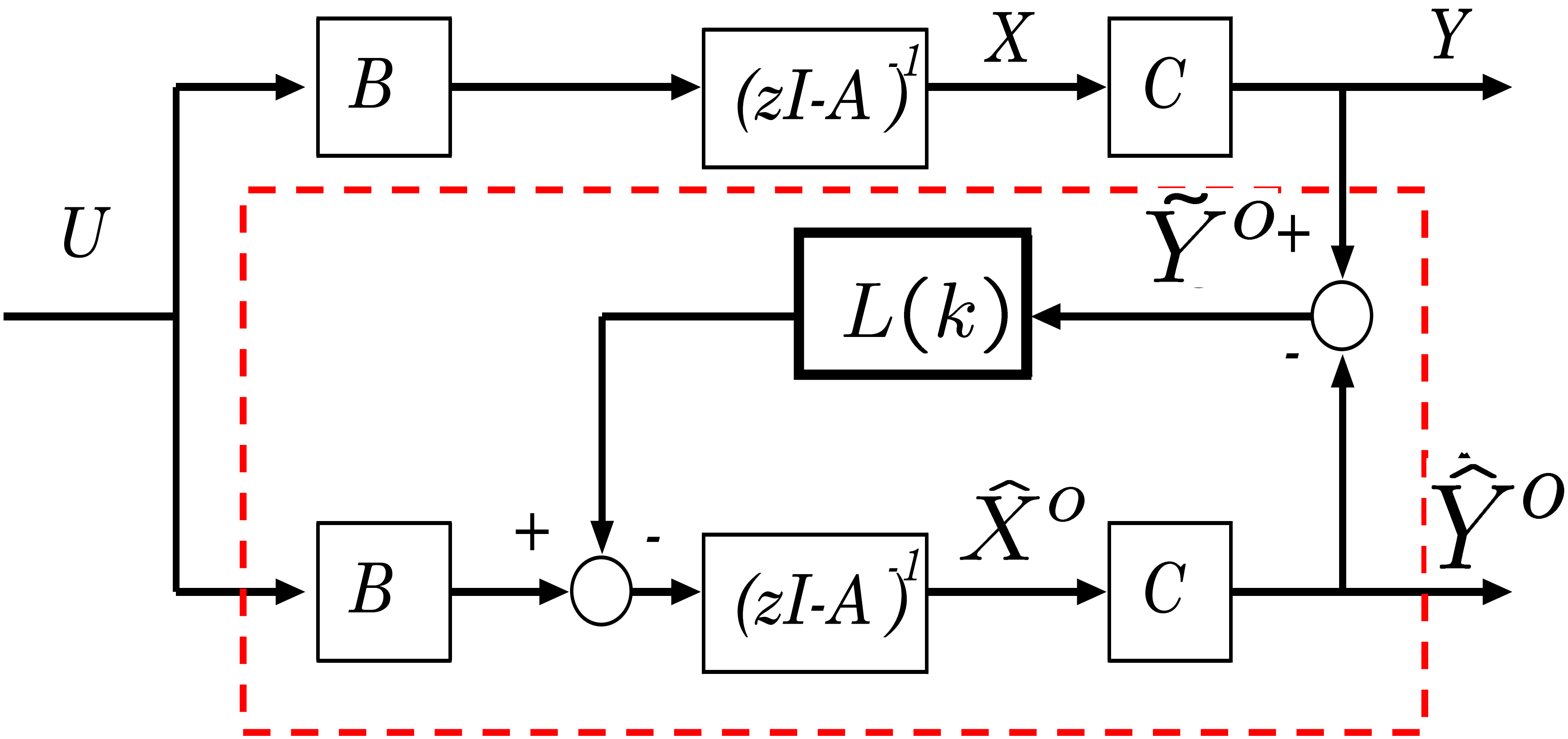
A-priori state observer structure:

$$\begin{aligned} \widehat{x}^o(k+1) &= A\,\widehat{x}^o(k) + B\,u(k) + L(k)\,\widetilde{y}^o(k) \\ \widetilde{y}^o(k) &= y(k) - C\,\widehat{x}^o(k) \end{aligned}$$

$$\begin{aligned} L(k) &= A\,M(k)C^T\left[C\,M(k)C^T + V(k)\right]^{-1} \\ M(k+1) &= AM(k)A^T + B_w W(k) B_w^T \\ &\quad - AM(k)C^T\left[CM(k)C^T + V(k)\right]^{-1} CM(k)A^T \\ M(0) &= X_o \end{aligned}$$

Kalman Filter Solution V-2 (review)

- Same structure as deterministic a-priori observer



Kalman Filter State Space (review)

$$\hat{x}^o(k+1) = [A - L(k)C]\hat{x}^o(k) + \begin{bmatrix} B & L(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\hat{x}(k) = [I - F(k)C]\hat{x}^o(k) + \begin{bmatrix} 0 & F(k) \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\begin{aligned}
F(k) &= M(k)C^T \left[C\, M(k)C^T + V(k) \right]^{-1} \\
L(k) &= AM(k)C^T \left[C\, M(k)C^T + V(k) \right]^{-1} \\
M(k+1) &= AM(k)A^T + B_w W(k) B_w^T \\
&\quad - AM(k)C^T \left[CM(k)C^T + V(k) \right]^{-1} CM(k)A^T
\end{aligned}$$

Kalman Filter (KF) Properties (review)

The KF a-priori output error (*a-priori output residual*)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is often called the ***innovation***

it contains only the "new information" in $y(k)$

Moreover,

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,j) = [CM(k)C^T + V(k)]\delta(j)$$

i.e. $\tilde{y}^o(k)$ is an uncorrelated RVS

KF as an innovations filter (review)

For the figure on the next slide, we will assume without loss of generality that the control input is zero, i.e.

$$u(k) = 0 \qquad\qquad k = 0,\ 1,\ \cdots$$

- **Plant:**

$$\begin{aligned} x(k+1) &= A\,x(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

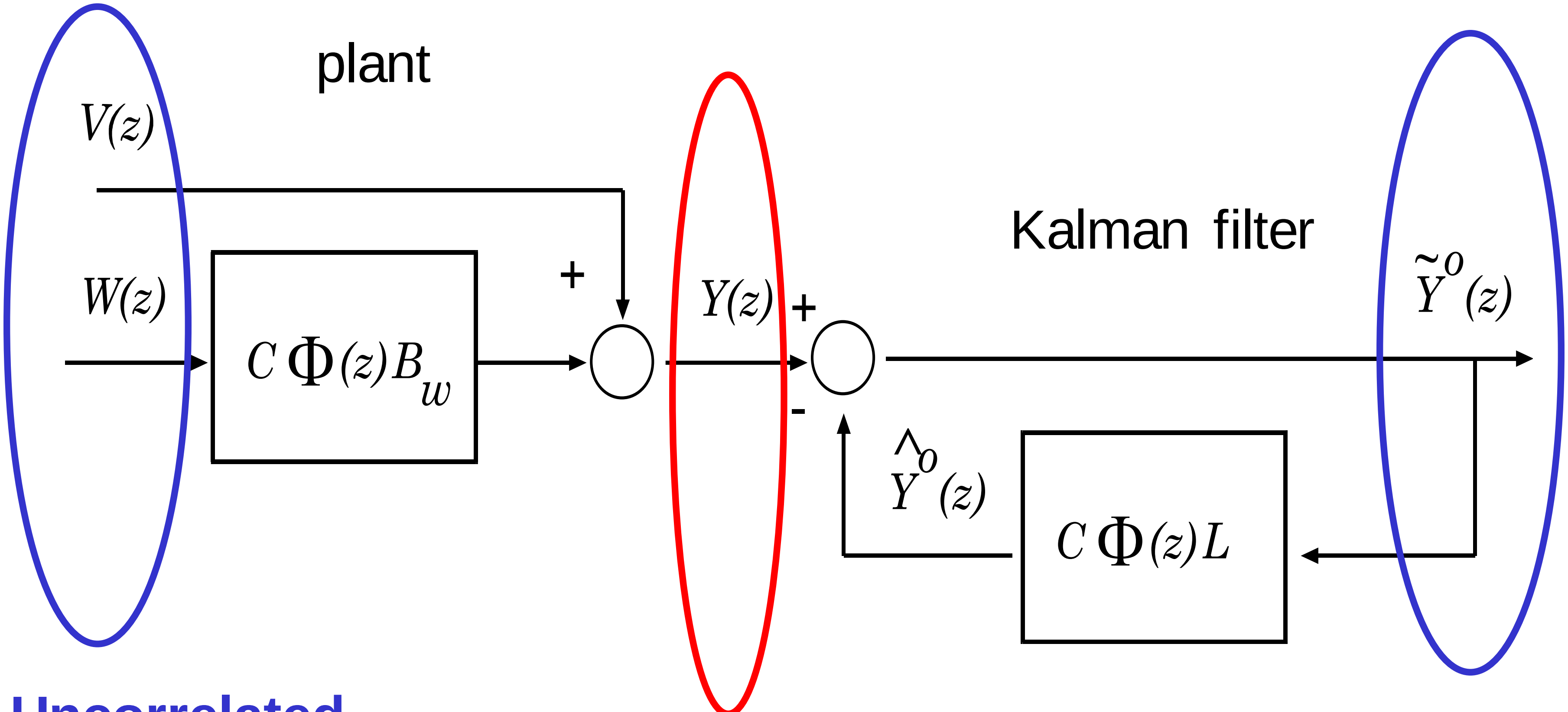
- **Kalman filter V-2:**

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + L(k)\tilde{y}^o(k)$$

$$\hat{y}^o(k) = C\hat{x}^o(k)$$

KF as an innovations filter (review)

$$\Phi(z) = (zI - A)^{-1}$$



Uncorrelated noise input

$$\begin{bmatrix} V(k) & 0 \\ 0 & W(k) \end{bmatrix}$$

Correlated noise output

Uncorrelated noise output

$$CM(k)C^T + V(k)$$

Kalman Filter (KF) Properties (review)

- The KF is a linear time varying estimator.
- The KF is the **optimal state estimator** when the input and measurement noises are Gaussian.
- The KF is still the **optimal *linear* state estimator** even when the input and measurement noises are **not** Gaussian.
- The KF covariance Riccati equation is iterated in a forward manner, rather than in a backwards manner as in the LQR.

$$M(0) \rightarrow M(k)$$

Steady State Kalman Filter

- Assume now that we want to estimate the state under zero-mean, stationary input and output Gaussian white noise, I.e.

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$
$$y(k) = Cx(k) + v(k)$$

$$\left.\begin{aligned} E\{w(k)\} &= 0 \\ E\{v(k)\} &= 0 \\ E\{w(k+l)w^T(k)\} &= W\,\delta(l) \\ E\{v(k+l)v^T(k)\} &= V\,\delta(l) \\ E\{w(k+l)v^T(k)\} &= 0 \end{aligned}\right\}$$

WSS Gaussian Noise

A priori estimation error dynamics

$$\tilde{x}^o(k+1) = [A - L(k)C]\tilde{x}^o(k) + B_w w(k) - L(k)v(k)$$

Proof:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_w w(k) \\ \hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)\tilde{y}^o(k) \end{cases}$$

Subtracting equations gives

$$\tilde{x}^o(k+1) = A\tilde{x}^o(k) + B_w w(k) - L(k)\tilde{y}^o(k)$$

$$C\tilde{x}^o(k) + v(k)$$

$\blacksquare$

Steady state Kalman filter, question 1

1) When does there exist a **BOUNDED limiting** solution

$$M_\infty$$

to the Riccati Eq.

$$M(k+1) = AM(k)A^T + B_w W B_w^T - AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T$$

for each choice of $M(0) \succeq 0$?

Steady state Kalman filter, question 2

2) When does there exist a **UNIQUE limiting** solution

$$M_\infty$$

to the Riccati Eq.

$$\begin{aligned} M(k+1) = {} & AM(k)A^T + B_w W B_w^T \\ & - AM(k)C^T[CM(k)C^T + V]^{-1} CM(k)A^T \end{aligned}$$

regardless of the choice of $M(0) \succeq 0$?

Steady state Kalman filter, question 3

3) When does the **limiting** solution

$$M_\infty$$

to the Riccati Eq.

yield **asymptotically stable** estimation error dynamics?

$$A_c = A - L_\infty C$$

is Schur
(all eigenvalues inside unit circle)

$$L_\infty = A M_\infty C^T \left[C M_\infty C^T + V \right]^{-1}$$

Detectability Assumption

We are only interested in the case where the estimation error dynamics are asymptotically stable

If *(C,A)* is not detectable, then there does not exist a estimator that results is asymptotically stable estimation error dynamics

$\Rightarrow$ For the stationary Kalman filter, we always assume that *(C,A)* is detectable

Theorem 1 : Existence of a bounded $\boldsymbol{M}_\infty$

Let (C, A) be detectable
(unobservable modes are asymptotically stable)

Then, for $M(0) = X_0 = 0$ as $k \to \infty$
the solution of the Riccati Eq.

$$M(k+1) = AM(k)A^T + B_w W B_w^T \\ - AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T$$

converges to a **BOUNDED limiting** solution $\boldsymbol{M}_\infty$
that satisfies the algebraic Riccati equation (DARE):

$$M_\infty = AM_\infty A^T + B_w W B_w^T \\ - AM_\infty C^T[CM_\infty C^T + V]^{-1}CM_\infty A^T$$

Theorem 1 : Notes

- Theorem 1 only guarantees the existence of a bounded solution $\boldsymbol{M}_\infty$ to the algebraic Riccati Equation

$$M_\infty = AM_\infty A^T + B_w W B_w^T - AM_\infty C^T [CM_\infty C^T + V]^{-1} CM_\infty A^T$$

- The solution may not be unique.

- Different initial conditions $M(0) = X_0$ may result in different limiting solutions $\boldsymbol{M}_\infty$ or may yield no limiting solution at all!

Theorem 2 : Existence and uniqueness of a positive definite asymptotic stabilizing solution

If *(C,A)* is detectable and *(A,$B_w W^{1/2}$)* is controllable

1) There exists a unique, bounded solution $M_\infty \succ 0$ to the DARE

$$M_\infty = A M_\infty A^T + B_w W B_w^T \\ - A M_\infty C^T [C M_\infty C^T + V]^{-1} C M_\infty A^T$$

2) The estimation error dynamics are **<u>asymptotically stable</u>**

$$\tilde{x}^o(k+1) = [A - L_\infty C]\tilde{x}^o(k) + B_w w(k) - L_\infty v(k)$$

$$L_\infty = A M_\infty C^T [C M_\infty C^T + V]^{-1}$$

Theorem 3 : Existence of a stabilizing solution

If *(C,A)* is detectable and *(A,$B_w W^{1/2}$)* is stabilizable

1) There exists a unique, bounded solution $M_\infty \succeq 0$ to the DARE

$$M_\infty = AM_\infty A^T + B_w W B_w^T - AM_\infty C^T [CM_\infty C^T + V]^{-1} CM_\infty A^T$$

2) The estimation error dynamics are **asymptotically stable**

$$\tilde{x}^o(k+1) = [A - L_\infty C]\tilde{x}^o(k) + B_w w(k) - L_\infty v(k)$$

$$L_\infty = AM_\infty C^T [CM_\infty C^T + V]^{-1}$$

Theorem 4: A different approach

The discrete algebraic Riccati equation (DARE) has a solution for which $A - L_\infty C$ is Schur

if and only if

(C, A) is detectable and the matrix pair $(A,\ B_w W^{1/2})$ has no uncontrollable modes on the unit circle.

$$L_\infty = A M_\infty C^T [C M_\infty C^T + V]^{-1}$$

$$\begin{aligned} M_\infty = {} & A M_\infty A^T + B_w W B_w^T \\ & - A M_\infty C^T [C M_\infty C^T + V]^{-1} C M_\infty A^T \end{aligned}$$

Kalman Filter Solution V-1

A-posteriori state observer structure:

$$\begin{aligned} \hat{x}(k) &= \hat{x}^o(k) + F\tilde{y}^o(k) \\ \hat{x}^o(k+1) &= A\,\hat{x}(k) + B\,u(k) \\ \tilde{y}^o(k) &= y(k) - C\,\hat{x}^o(k) \end{aligned}$$

$$\begin{aligned} F &= MC^T\left[C\,MC^T + V\right]^{-1} \\ M &= AMA^T + B_w W B_w^T \\ &\quad - AMC^T(CMC^T + V)^{-1}CMA^T \end{aligned}$$

$A - AFC$ is Schur

Kalman Filter Solution V-2

A-priori state observer structure:

$$\hat{x}^o(k+1) \;=\; A\hat{x}^o(k) + Bu(k) + L\tilde{y}^o(k)$$

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

$$L \;=\; AMC^T\left[C\,MC^T + V\right]^{-1}$$

$$\begin{aligned} M \;=\; & AMA^T + B_w W B_w^T \\ & - AMC^T(CMC^T + V)^{-1}CMA^T \end{aligned}$$

$A - LC$ is Schur

Kalman Filter State Space

$$\hat{x}^o(k+1) = [A - LC]\hat{x}^o(k) + \begin{bmatrix} B & L \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\hat{x}(k) = [I - FC]\hat{x}^o(k) + \begin{bmatrix} 0 & F \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}$$

$$\begin{aligned} F &= MC^T \left[C\,MC^T + V\right]^{-1} \\ L &= AMC^T \left[CMC^T + V\right]^{-1} \\ M &= AMA^T + B_w W B_w^T \\ &\quad - AMC^T(CMC^T + V)^{-1}CMA^T \end{aligned}$$

$A - LC$ is Schur

Kalman Filter (KF) Properties

The KF a-priori output error (*a-priori output residual*)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is often called the ***innovation***

it contains only the "new information" in $y(k)$

Moreover,

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(j) = [CMC^T + V]\delta(j)$$

i.e. $\tilde{y}^o(k)$ is white

KF as an innovations filter

For the figure on the next slide, we will assume without loss of generality that the control input is zero, i.e.

$$u(k) = 0 \qquad\qquad k = 0,\, 1,\, \cdots$$

- **Plant:**

$$\begin{aligned} x(k+1) &= A\,x(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

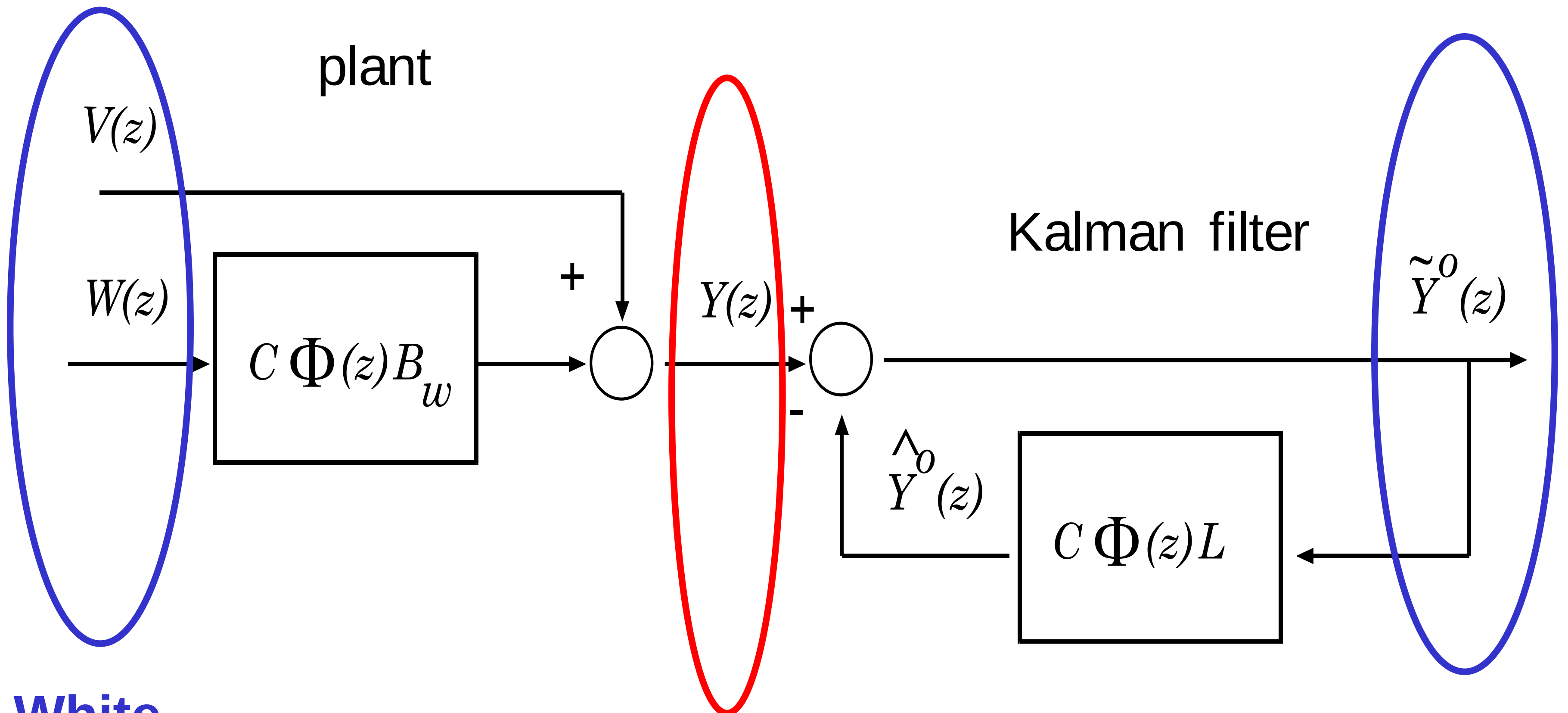
- **Kalman filter V-2:**

$$\widehat{x}^o(k+1) = A\widehat{x}^o(k) + L\tilde{y}^o(k)$$

$$\widehat{y}^o(k) = C\widehat{x}^o(k)$$

KF as an innovations (whitening) filter

$$\Phi(z) = (zI - A)^{-1}$$



White noise input

$$\begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix}$$

Colored noise output

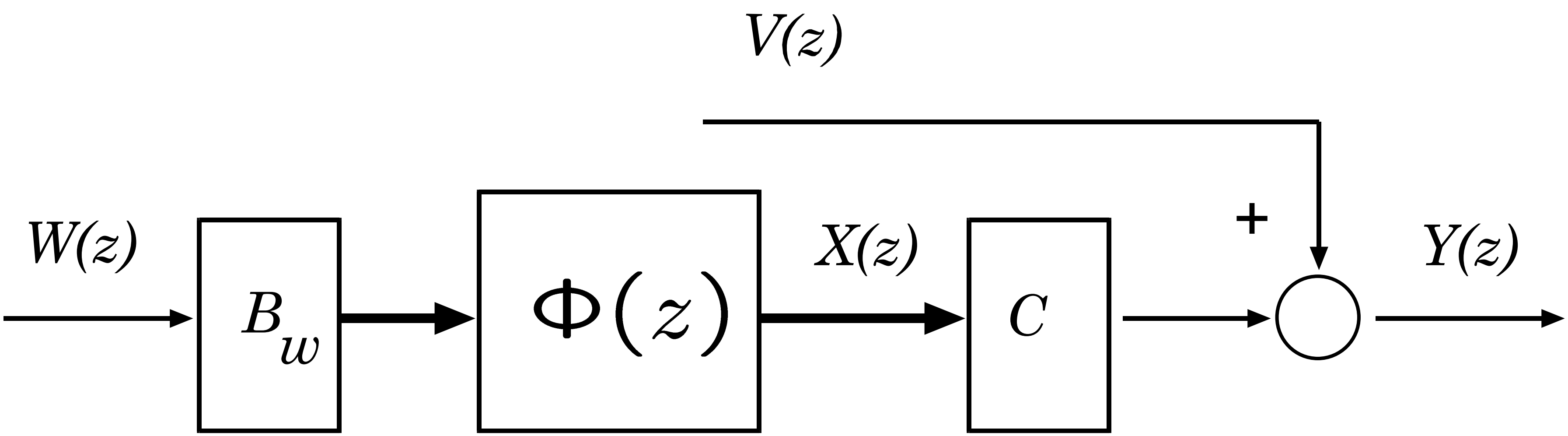
White noise output

$$CMC^T + V$$

Output $\boldsymbol{Y(k)}$ is colored noise

- **Plant:**

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B_w\,w(k) \\
y(k) &= C\,x(k) + v(k)
\end{aligned}
$$



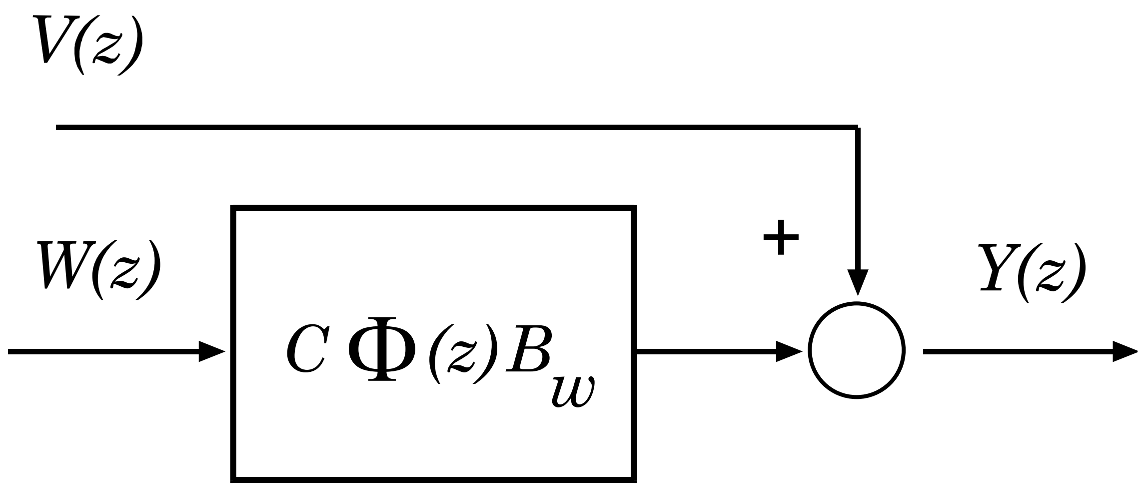
$$\Phi(z) = (zI - A)^{-1}$$

Output *Y(k)* is colored noise

- **Plant:**

$$Y(z) = [C\Phi(z)B_w]\ W(z) + V(z)$$

$$\Phi(z) = (zI - A)^{-1}$$

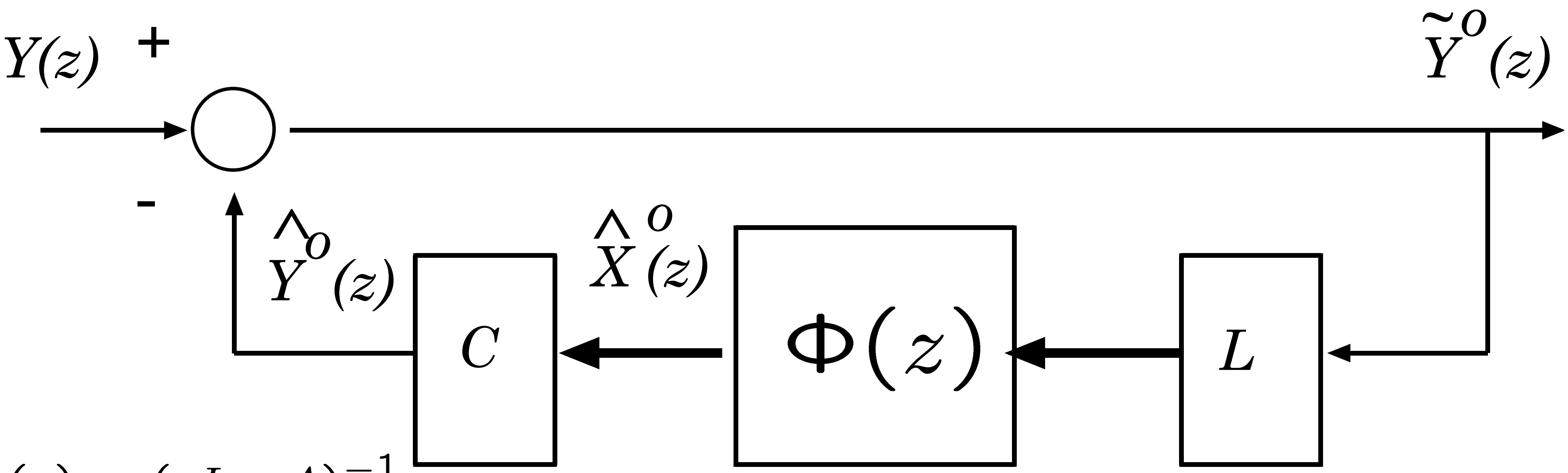


KF as an innovations filter

- **A-priori KF:**

$$\hat{x}^o(k+1) = A\,\hat{x}^o(k) + L\,\tilde{y}^o(k)$$

$$\tilde{y}^o(k) = y(k) - C\,\hat{x}^o(k)$$



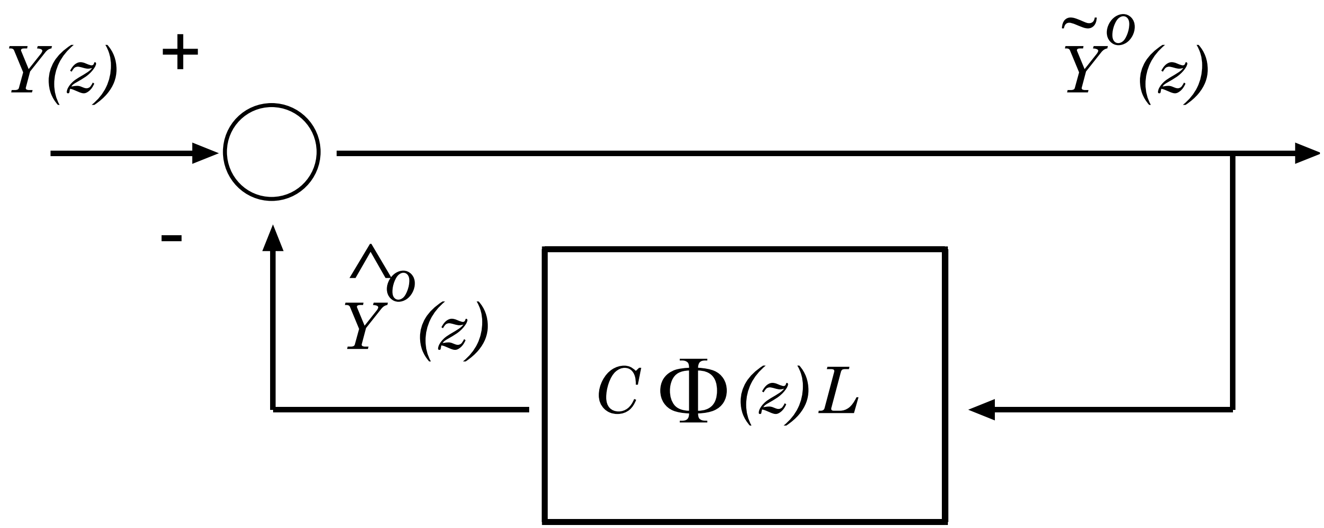
$$\Phi(z) = (zI - A)^{-1}$$

KF as an innovations filter

- **A-priori KF:**

$$\tilde{Y}^o(z) \;=\; [I + C\Phi(z)L]^{-1}\,Y(z)$$

$$\Phi(z) = (zI - A)^{-1}$$

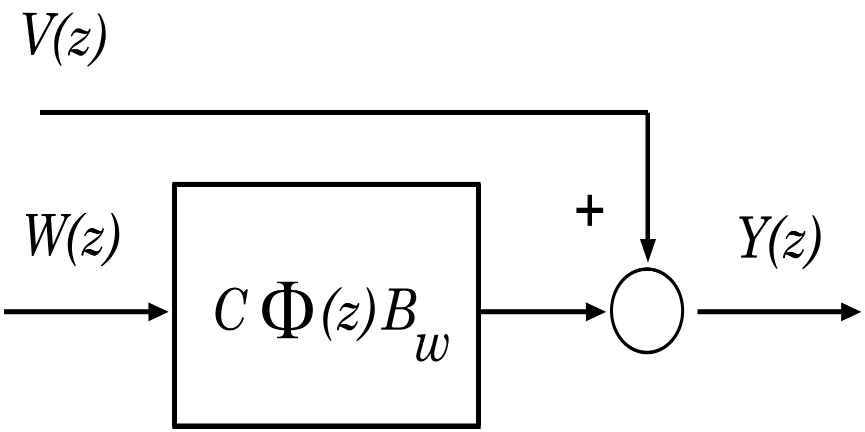


KF as an innovations filter

- **Plant** $$Y(z) = [C\Phi(z)B_w]\ W(z) + V(z)$$

- **A-priori KF:**

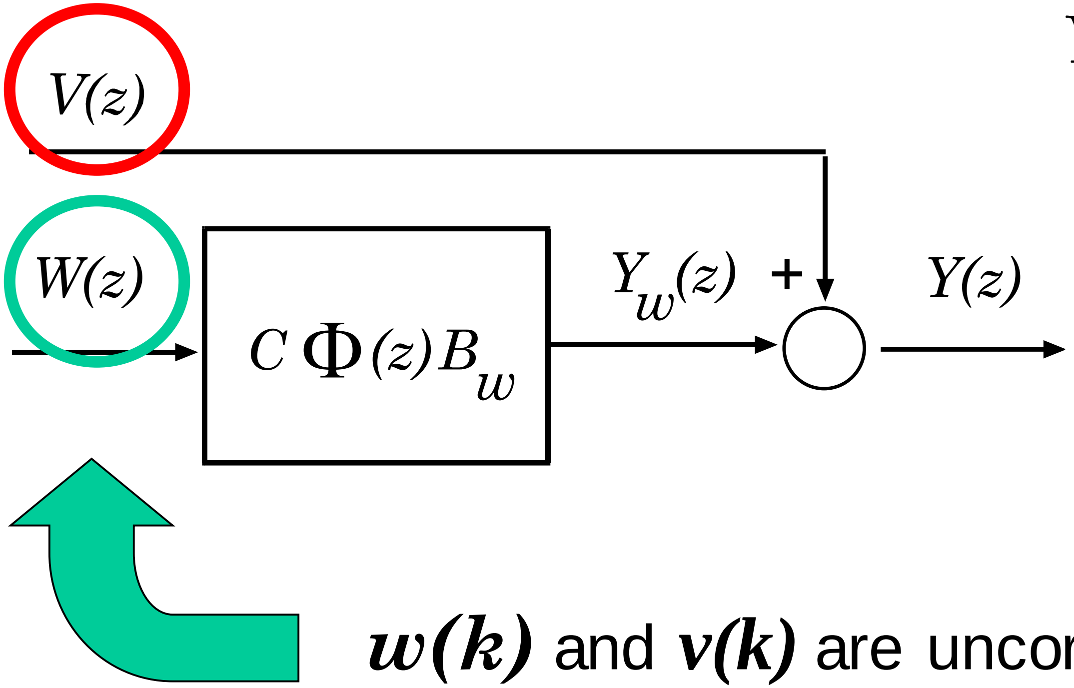
$$\tilde{Y}^o(z) \;=\; [I + C\Phi(z)L]^{-1}\,Y(z)$$



Y(k) Power spectrum

Power spectrum of $y(k)$

$$Y(z) = \underbrace{[C\Phi(z)B_w]\, W(z)}_{Y_w(z)} + V(z)$$

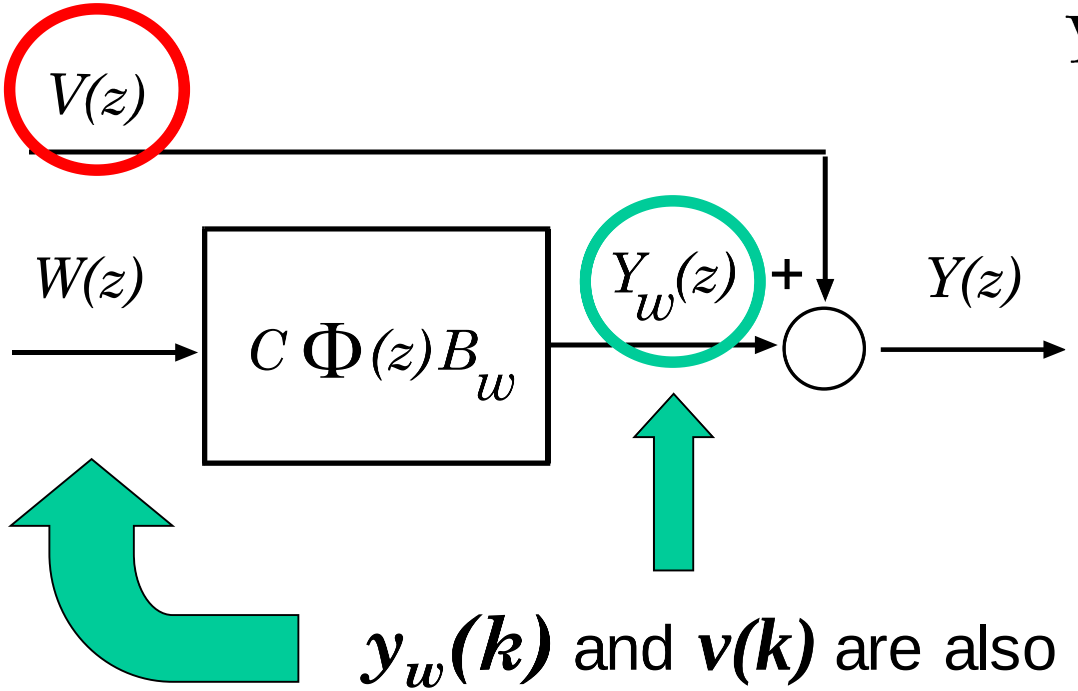


w(k) and ***v(k)*** are uncorrelated!

Y(k) Power spectrum

Power spectrum of $y(k)$

$$Y(z) = \underbrace{[C\Phi(z)B_w]\,W(z)}_{Y_w(z)} + V(z)$$

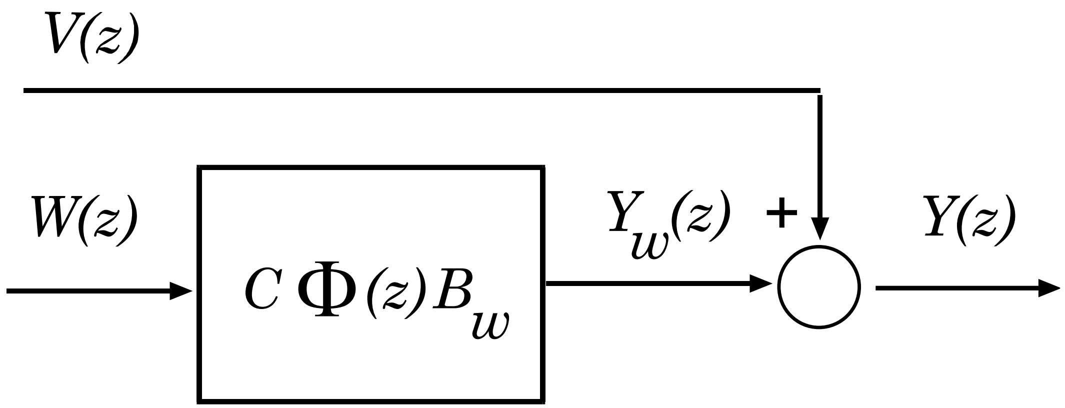


$y_w(k)$ and $v(k)$ are also uncorrelated!

Y(k) Power spectrum

Power spectrum of $y(k)$

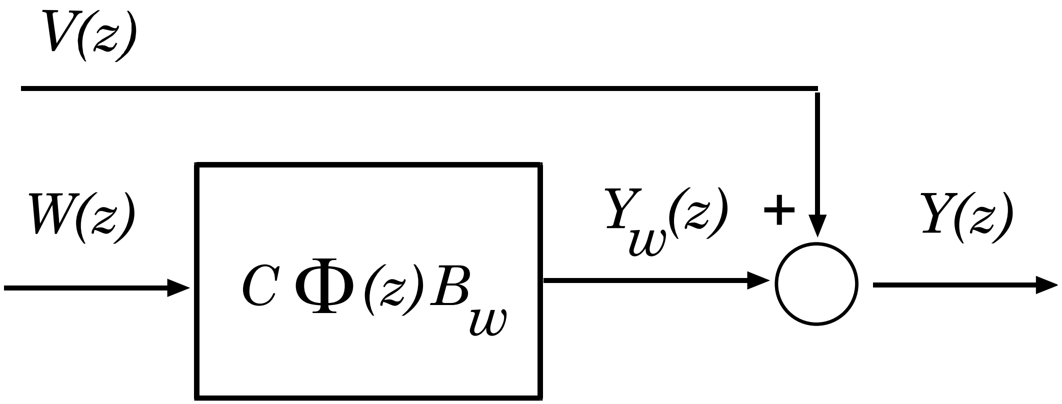
$$Y(z) = \underbrace{[C\Phi(z)B_w]\ W(z)}_{Y_w(z)} + V(z)$$



$$\Lambda_{yy}(z) = \Lambda_{y_w y_w}(z) + \Lambda_{vv}(z)$$

Y(k) Power spectrum

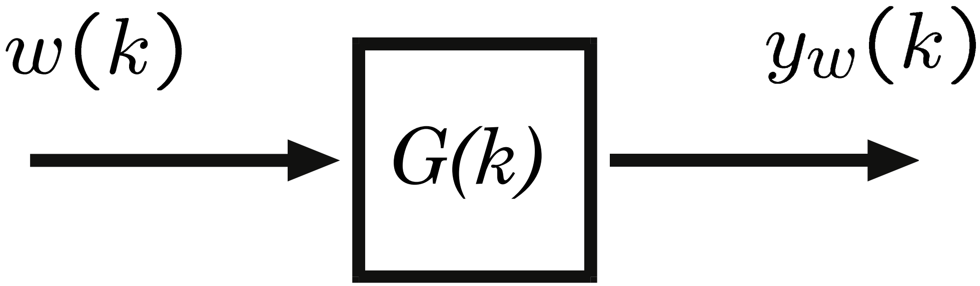
Power spectrum of $y(k)$



$$\Lambda_{yy}(z) = \Lambda_{y_w y_w}(z) + \underbrace{\Lambda_{vv}(z)}_{V}$$

v(k) is white noise

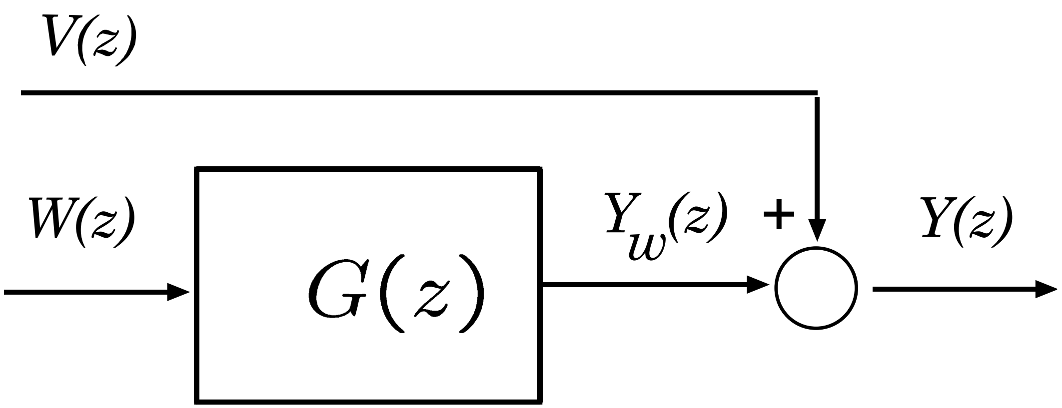
Y(k) Power spectrum



$$\Lambda_{y_w y_w}(z) = G(z)\,\Lambda_{ww}(z)\,G^T(z^{-1})$$

Y(k) Power spectrum

Power spectrum of $y(k)$



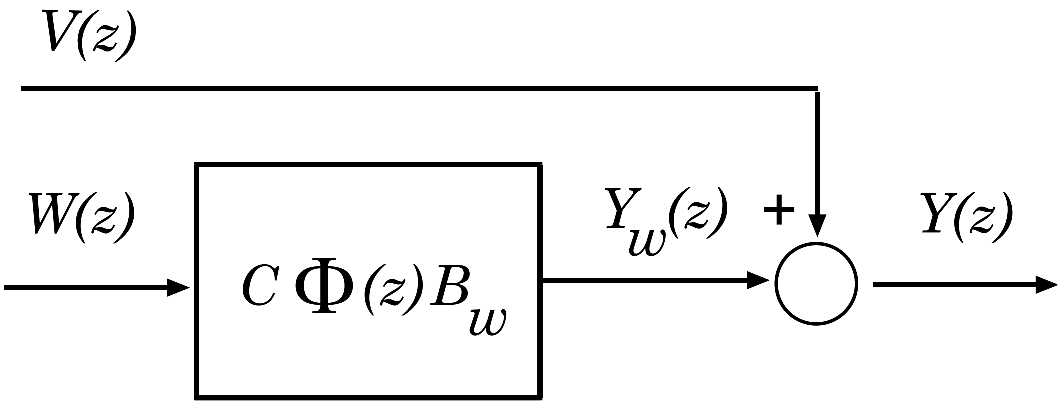
$$\Lambda_{yy}(z) = \underbrace{\Lambda_{y_w y_w}(z)} + V$$

$$\Lambda_{y_w y_w}(z) = [C\Phi(z)B_w]\underbrace{W}\left[C\Phi(z^{-1})B_w\right]^T$$

w(k) is white noise

Y(k) Power spectrum

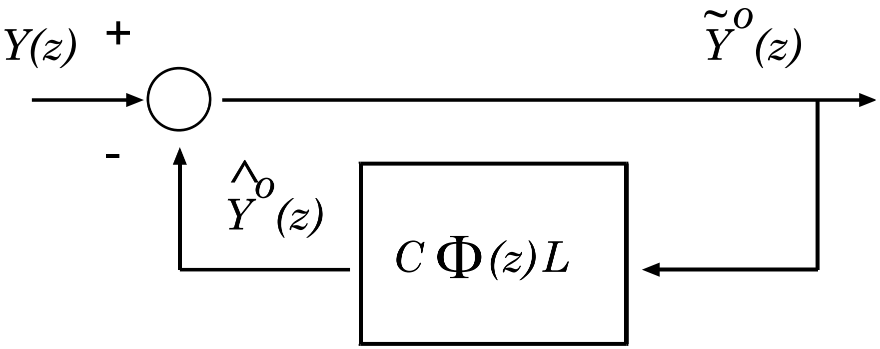
Power spectrum of $y(k)$



$$\Lambda_{yy}(z) = V + [C\Phi(z)B_w]\, W \left[C\Phi(z^{-1})B_w\right]^T$$

KF as an innovations filter

Power spectrum of $\tilde{y}^o(k)$



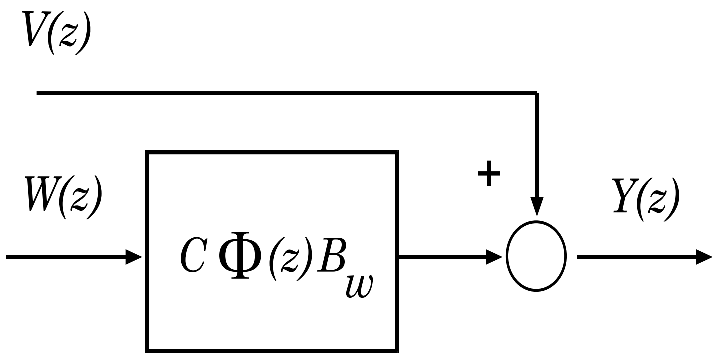
$$\tilde{Y}^o(z) \;=\; [I + C\Phi(z)L]^{-1}\, Y(z)$$

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(z) = [I + C\Phi(z)L]^{-1}\;\Lambda_{yy}(z)\left[I + C\Phi(z^{-1})L\right]^{-T}$$

$$\Lambda_{yy}(z) = [I + C\Phi(z)L]\;\Lambda_{\tilde{y}^o\tilde{y}^o}(z)\left[I + C\Phi(z^{-1})L\right]^{T}$$

KF as an innovations filter

Combining two results:



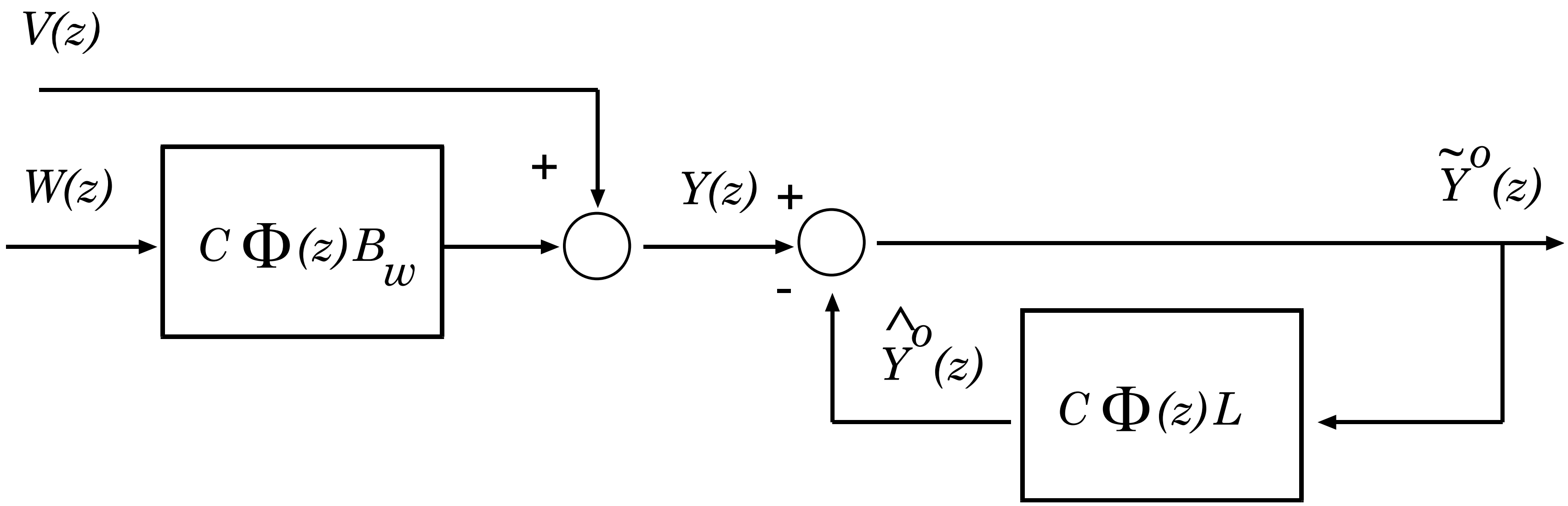
$$\Lambda_{yy}(z) = V + [C\Phi(z)B_w]\; W \left[C\Phi(z^{-1})B_w\right]^T$$

and

$$\Lambda_{yy}(z) = [I + C\Phi(z)L]\; \Lambda_{\tilde{y}^o\tilde{y}^o}(z) \left[I + C\Phi(z^{-1})L\right]^T$$

KF as an innovations filter

Combining two results:



$$[I + C\Phi(z)L]\ \Lambda_{\tilde{y}^o\tilde{y}^o}(z)\left[I + C\Phi(z^{-1})L\right]^T =$$

$$V + [C\Phi(z)B_w]\ W\left[C\Phi(z^{-1})B_w\right]^T$$

KF as an innovations filter

Recall what Theorem part 3) says about the a-priori output error (the innovation sequence)

$$
\begin{aligned}
\Lambda_{\tilde{y}^o \tilde{y}^o}(l) &= E\left\{\tilde{y}^o(k+l)\tilde{y}^{oT}(k)\right\} \\
&= \left[C\,MC^T + V\right]\delta(l)
\end{aligned}
$$

$\tilde{y}^o(k)$ is also white noise!!

KF as an innovations filter

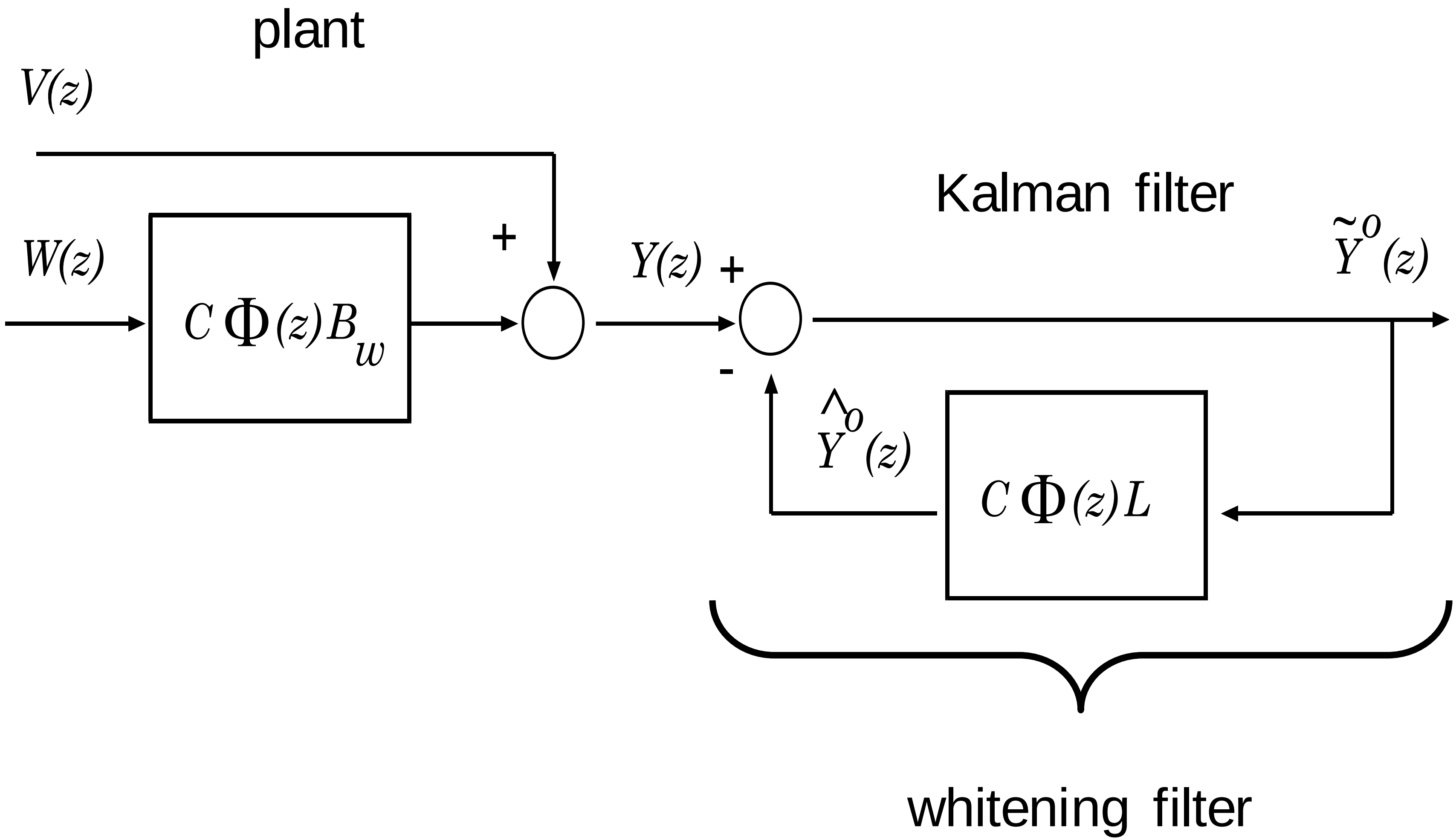
Recall what Theorem part 3) says about the a-priori output error (the innovation sequence)

$$\Lambda_{\tilde{y}^o \tilde{y}^o}(l) \;=\; \left[C\, M C^T + V\right] \delta(l)$$

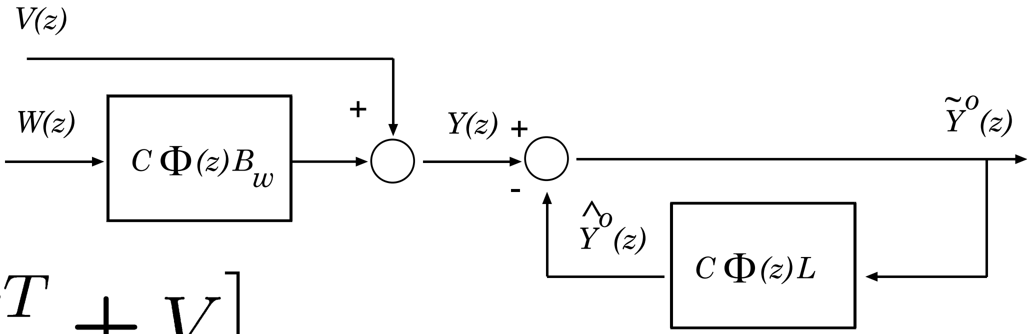
$$\boxed{\begin{aligned} \Lambda_{\tilde{y}^o \tilde{y}^o}(z) \;&=\; \left[C\, M C^T + V\right] \\ \Phi_{\tilde{y}^o \tilde{y}^o}(\omega) \;&=\; \left[C\, M C^T + V\right] \end{aligned}}$$

KF as a innovations (whitening) filter

•



KF as an innovations filter



$$\Lambda_{\tilde{y}^o\tilde{y}^o}(z) \;=\; \left[C\,MC^T + V\right]$$

therefore,

$$[I + C\Phi(z)L]\;\underbrace{\left[C\,MC^T + V\right]}_{=\Lambda_{\tilde{y}^o\tilde{y}^o}(z)}\left[I + C\Phi(z^{-1})L\right]^T =$$

$$\underbrace{V + [C\Phi(z)B_w]\;W\left[C\Phi(z^{-1})B_w\right]^T}_{=\Lambda_{yy}(z)}$$

KF return difference equality

$$[I + C\Phi(z)L] \left[C\, MC^T + V\right] \left[I + C\Phi(z^{-1})L\right]^T =$$

$$V + [C\Phi(z)B_w]\; W \left[C\Phi(z^{-1})B_w\right]^T$$

Kalman Filter & LQR Duality

Recall Steady state LQR:

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$

$$u(k) = -K\,x(k) + r(k)$$

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k)\, C_Q^T\, C_Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

$$Q = C_Q^T\, C_Q \geq 0 \qquad\qquad R = R^T > 0$$

Note:

We need to distinguish between:

- **LQR:** state cost weight $Q = C_Q^T C_Q \geq 0$

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k)\, C_Q^T\, C_Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

- **KF:** output matrix C

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

Kalman Filter & LQR Duality

Infinite-horizon LQR Closed-loop dynamics:

$$x(k+1) \;=\; (A - B\,K)\,x(k) + B\,r(k)$$

$$K = \left[R + B^T P B\right]^{-1} B^T P A$$

$$\begin{aligned} A^T P A - P \;&=\; -C_Q^T C_Q \\ &\quad + A^T P B \left[B^T P B + R\right]^{-1} B^T P A \end{aligned}$$

Kalman Filter & LQR Duality

Steady State KF Estimation error dynamics

$$\tilde{x}^o(k+1) \;=\; (A - LC)\,\tilde{x}^o(k) + B_w\, w(k) - Lv(k)$$

$$L \;=\; A\,MC^T\left[C\,MC^T + V\right]^{-1}$$

$$\begin{aligned} AMA^T - M \;&=\; -B_w W B_w^T \\ &\quad + AMC^T\left[CMC^T + V\right]^{-1} CMA^T \end{aligned}$$

Kalman Filter & LQR Duality

Let's compare the DAREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T$$

$$\boxed{P \Rightarrow M}$$

Kalman Filter & LQR Duality

Let's compare the AREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T$$

$$\boxed{C_Q^T \Rightarrow B_w W^{1/2} = B_w'}$$

Kalman Filter & LQR Duality

Let's compare the AREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T$$

$$\boxed{A \Rightarrow A^T}$$

Kalman Filter & LQR Duality

Let's compare the AREs:

LQR

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R\right]^{-1} B^T P A$$

KF

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V\right]^{-1} C M A^T$$

$$B \Rightarrow C^T$$

Kalman Filter & LQR Duality

Let's compare the AREs:

$$A^T P A - P = -C_Q^T C_Q + A^T P B \left[B^T P B + R \right]^{-1} B^T P A$$

LQR

$$A M A^T - M = -B_w W B_w^T + A M C^T \left[C M C^T + V \right]^{-1} C M A^T$$

KF

$$\boxed{R \Rightarrow V}$$

Kalman Filter & LQR Duality

Let's compare the Feedback gains:

$$K = \left[R + B^T P B\right]^{-1} B^T P A$$

$\boxed{LQR}$

$$L^T = \left[V + C M C^T\right]^{-1} C M A^T$$

$\boxed{KF}$

$P \Rightarrow M \qquad B \Rightarrow C^T \qquad A \Rightarrow A^T \qquad R \Rightarrow V$

Kalman Filter & LQR Duality

Let's compare the Feedback gains:

$$K^T = APB\left[R + B^T PB\right]^{-1}$$

LQR

$$L = AMC^T\left[V + CMC^T\right]^{-1}$$

KF

$$\boxed{K^T \Rightarrow L}$$

Kalman Filter & LQR Duality

Comparing ARE's and feedback gains, we obtain the following duality

duality $\longrightarrow$

LQR	KF
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A\text{-}BK)$	$(A\text{-}LC)^T$

Kalman Filter & LQR Duality

$\xrightarrow{\textit{duality}}$

LQR	**KF**
P	M
A	A^T
B	C^T
R	V
${C_Q}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A\text{-}BK)$	$(A\text{-}LC)^T$

$$A^T P A - P + C_Q^T C_Q - A^T P B \left[B^T P B + R \right]^{-1} B^T P A = 0$$

$$A M A^T - M + B'_w B'^T_w - A M C^T \left[C M C^T + V \right]^{-1} C M A^T = 0$$

Kalman Filter & LQR Duality

duality →

LQR	KF
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A\text{-}BK)$	$(A\text{-}LC)^T$

$$K = \left[B^T P B + R\right]^{-1} B^T P A$$

$$L^T = \left[C M C^T + V\right]^{-1} C M A^T$$

Kalman Filter & LQR Duality

- It is possible to use duality to prove theorems 1-4 for stationary Kalman filters from the corresponding theorems from the infinite horizon LQR

- The following slides give an outline of how to do this

- The main idea is to design an infinite horizon LQR for a fictitious system

Theorems 1-4 proof methodology

- Consider the LQR problem:

$$\bar{x}(k+1) = A^T \bar{x}(k) + C^T \bar{u}(k)$$

$$J = \bar{x}^T(N)\, X_0\, \bar{x}(N) + \sum_{k=0}^{N-1} \left\{ \bar{x}^T(k)\, B_w W B_w^T\, \bar{x}(k) + \bar{u}^T(k)\, V\, \bar{u}(k) \right\}$$

- Solution:

$$\bar{u}(k) = -[C\bar{P}(k+1)C^T + V]^{-1} C\bar{P}(k+1)A^T \bar{x}(k)$$

$$\begin{aligned} \bar{P}(k-1) = {} & A\bar{P}(k)A^T + B_w W B_w^T \\ & - A\bar{P}(k)C^T[C\bar{P}(k)C^T + V]^{-1} C\bar{P}(k)A^T \end{aligned}$$

$$\bar{P}(N) = X_0 \quad = M(0)$$

Theorems 1-4 proof methodology

- The solution of the Riccati equation

$$
\begin{aligned}
\bar{P}(k-1) &= A\bar{P}(k)A^T + B_w W B_w^T \\
&\quad - A\bar{P}(k)C^T[C\bar{P}(k)C^T + V]^{-1}C\bar{P}(k)A^T \\
\bar{P}(N) &= X_0 = M(0)
\end{aligned}
$$

is $\bar{P}(N-k) = M(k)$

- Use LQR convergence results for $\bar{P}(0)$ as $N \to \infty$ to yield convergence results for $\bar{M}(N)$ as $N \to \infty$

Theorems 1-4 proof methodology

- Other key ideas in proofs
  - (A^T, C^T) stabilizable iff (C, A) detectable
  - Unobservable modes of $((B_w W^{1/2})^T, A^T)$ are the uncontrollable modes of $(A, B_w W^{1/2})$
  - $A^T - C^T L^T$ is Schur iff $A - LC$ is Schur

Steady State LQR

Theorem 1):

If the pair ***[A, B]*** is controllable (or stabilizable), the solution of the DRE

$$
\begin{aligned}
-P(k) &= A^T P(k+1) A + C_Q^T C_Q \\
&\quad - A^T P(k+1) B \left[B^T P(k+1) B + R \right]^{-1} B^T P(k+1) A
\end{aligned}
$$

with $P(N) = 0$

converges, as $N \to \infty$, to a constant that satisfies

$$
P = A^T P A + C_Q^T C_Q - A^T P B \left[B^T P B + R \right]^{-1} B^T P A
$$

Steady State LQR

Theorem 2:

If the pair ***[A, C_q]*** is observable *(or detectable)*

*Then **[A,B]** is controllable (or stabilizable) if and only if:*

1) The solution of

$$-P(k) = A^T P(k+1) A + C_Q^T C_Q - A^T P(k+1) B \left[B^T P(k+1) B + R \right]^{-1} B^T P(k+1) A$$

with $P(N) \succeq 0$

*Converges to a **unique** stationary solution **P**, which satisfies*

$$P = A^T P A + C_Q^T C_Q - A^T P B \left[B^T P B + R \right]^{-1} B^T P A$$

Steady State LQ

Theorem 2: (continuation)

2) $\boldsymbol{P}$ is positive definite (*semi-definite*)

3) The close loop matrix $A_c = A - BK$ is **Schur**

$$K = \left[B^T P B + R\right]^{-1} B^T P A$$

Kalman Filter & LQR Duality

LQR	KF
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A\text{-}BK)$	$(A\text{-}LC)^T$

LQR ***KF***

$$[A, B]\ \text{controllable} \xrightarrow{\textit{duality}} [A^T, C^T]\ \text{controllable}$$

$$\big\downarrow$$

KF $[C, A]$ observable

Kalman Filter & LQR Duality

LQR	**KF**
P	M
A	A^T
B	C^T
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T
$(A\text{-}BK)$	$(A\text{-}LC)^T$

LQR ***KF***

$$[C_Q, A] \text{ observable} \xrightarrow{\textit{duality}} [B_w'^T, A^T] \text{ observable}$$

$$\downarrow$$

KF $[A, B_w'] \text{ controllable}$

Steady State Kalman Filter

Theorem 1:

If the pair ***[A,C]*** is observable (or detectable):

the solution of

$$M(k+1) \;=\; AM(k)A^T + B_w W B_w^T - AM(k)C^T\left[CM(k)C^T + V\right]^{-1} CM(k)A^T$$

with $M(0) = 0$

Converges to a stationary solution, ***M,*** which satisfies

$$M \;=\; AMA^T + B_w W B_w^T - AMC^T\left[CMC^T + V\right]^{-1} CMA^T$$

Steady State Kalman Filter

Theorem 2:

If the pair ***[A, B'$_w$]*** is controllable (*or stabilizable*), where

$$B_w^{'} = B_w W^{1/2}$$

*Then **[A,C]** is observable (or detectable) if and only if:*

1) The solution of

$$\begin{aligned} M(k+1) &= AM(k)A^T + B_w W B_w^T \\ &\quad - AM(k)C^T \left[CM(k)C^T + V\right]^{-1} CM(k)A^T \end{aligned}$$

$$M(0) \succeq 0$$

*Converges to a **unique** stationary solution **M**, which satisfies*

$$M = AMA^T + B_w W B_w^T - AMC^T \left[CMC^T + V\right]^{-1} CMA^T$$

Steady State Kalman Filter

Theorem 2: (continuation)

2) $\boldsymbol{M}$ is positive definite (*semi-definite*)

3) The close loop matrix $A_c = A - LC$ is **Schur**

$$L = AMC^T \left[CMC^T + V \right]^{-1}$$

Steady State Kalman Filter

Theorem 3:

Under stationary noise and the conditions in theorems 1) and 2),

The observer a-priori residual (innovations)

$$\tilde{y}^o(k) \;=\; y(k) - C\,\hat{x}^o(k)$$

is white

$$\boxed{E\left\{\tilde{y}^o(k+l)\tilde{y}^{oT}(k)\right\} = \left[C\,MC^T + V\right]\delta(l)}$$

KF as an innovations filter

We will assume, without loss of generality that the control input is zero, I.e.

$$u(k) = 0 \qquad\qquad k = 0,\, 1,\, \cdots$$

- **Plant:**

$$\begin{aligned} x(k+1) &= A\,x(k) + B_w\,w(k) \\ y(k) &= C\,x(k) + v(k) \end{aligned}$$

KF as an innovations (whitening) filter

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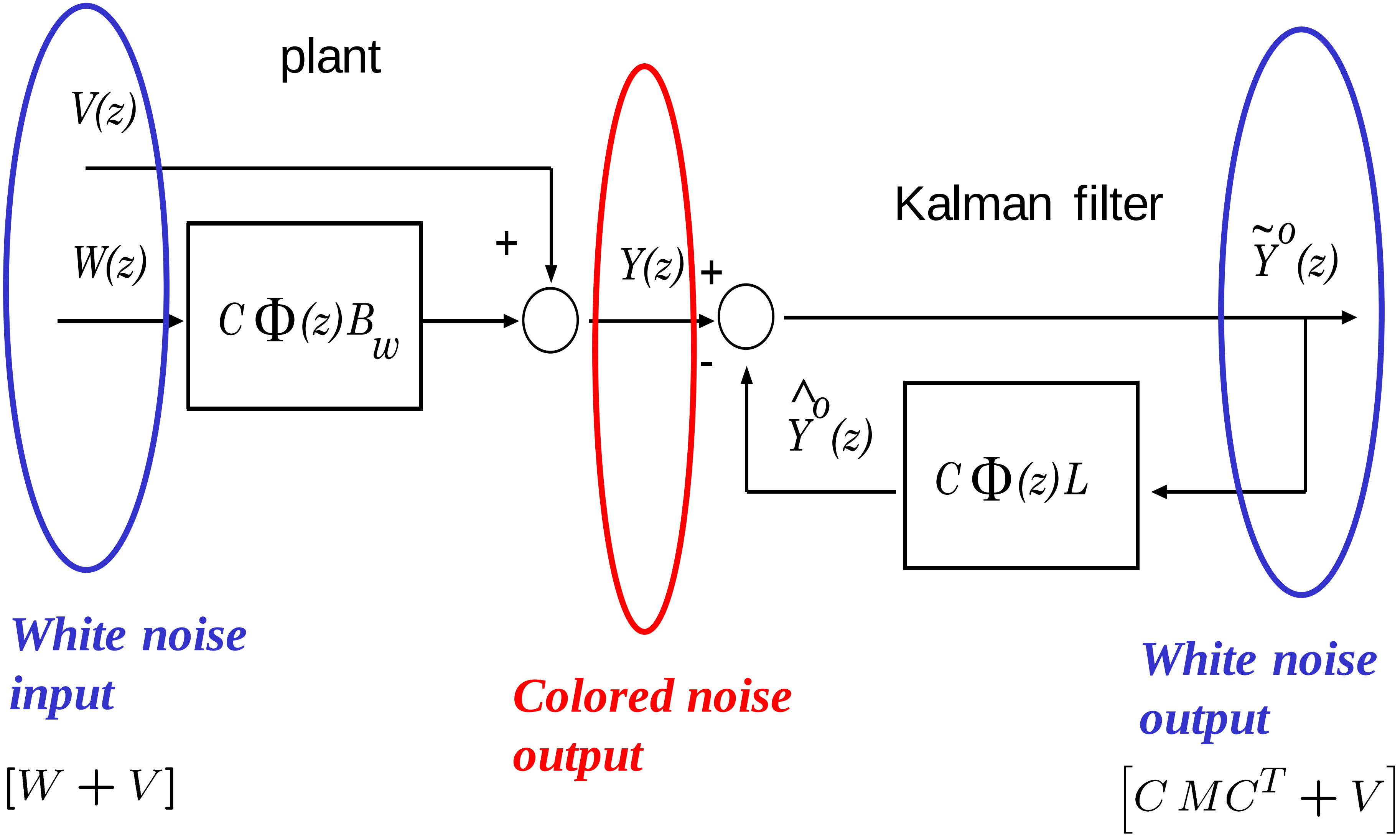
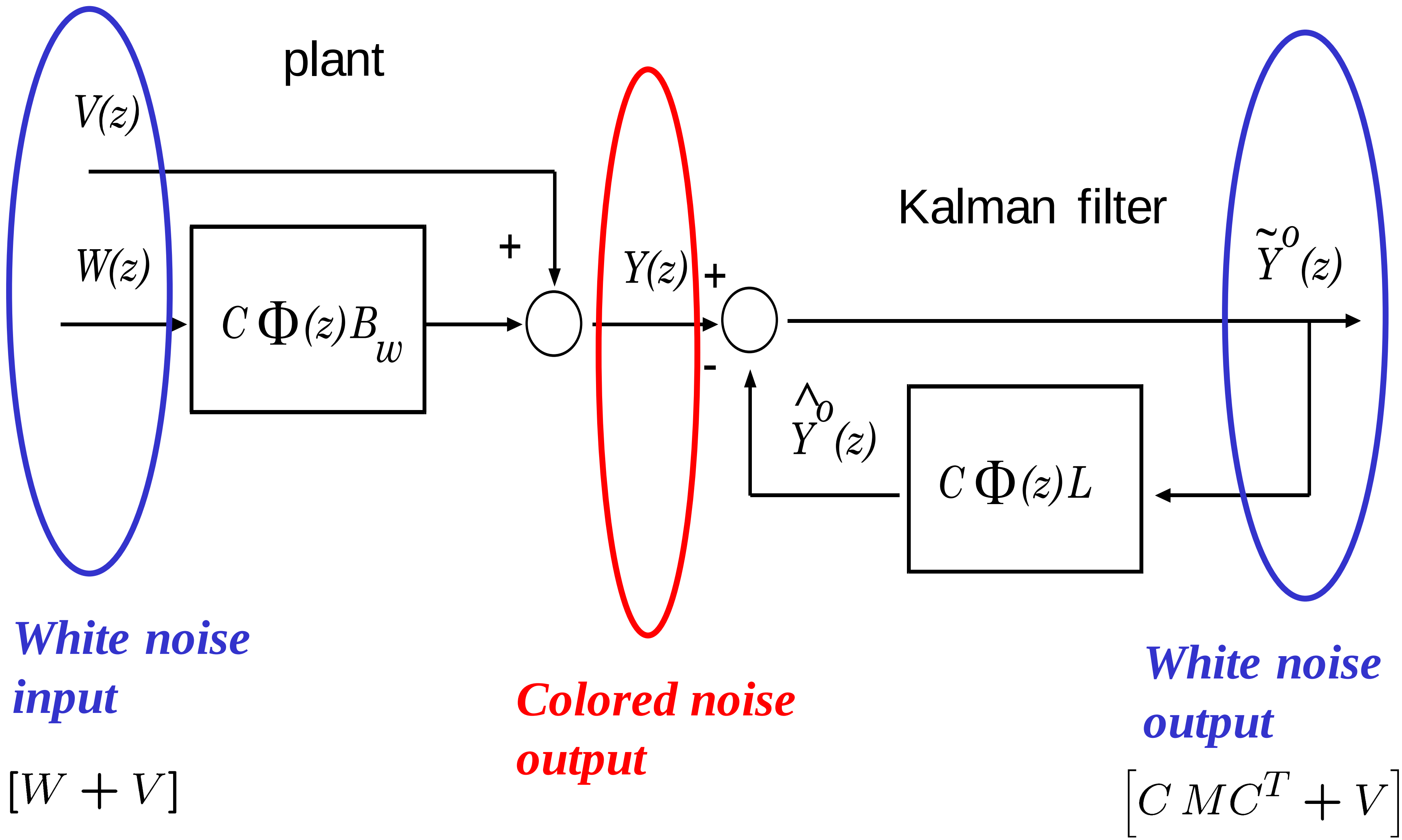
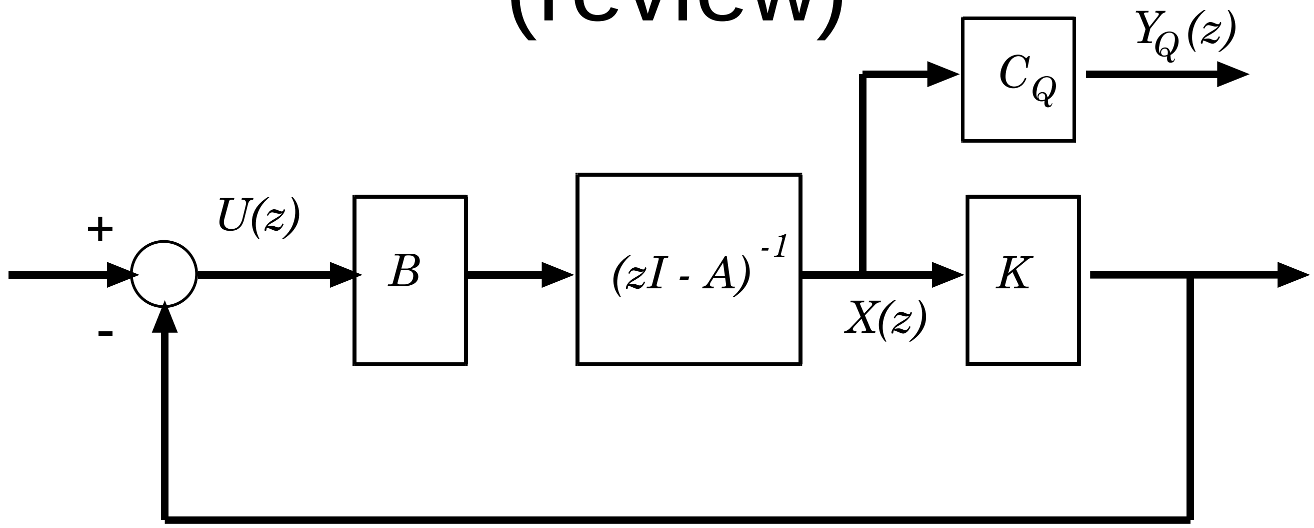


KF as a innovations (whitening) filter

•



Return difference equality for LQR (review)



$$[I + G_o(z^{-1})]^T \, [R + B^T P B] \, [I + G_o(z)] = R + G_Q^T(z^{-1}) \, G_Q(z)$$

Open loop transfer function:

$$G_o(z) = K \Phi(z) B$$

TF from $U(z)$ to $Y_Q(z)$:

$$G_Q(z) = C_Q \Phi(z) B$$

Return difference equality for LQR (review)

Substituting, $G_o(z) = K\Phi(z)B$ $G_Q(z) = C_Q\Phi(z)B$

into

$$[I + G_o(z^{-1})]^T [R + B^T P B] [I + G_o(z)] = R + G_Q^T(z^{-1}) G_Q(z)$$

We obtain,

$$\left[I + K\Phi(z^{-1})B\right]^T \left[B^T P B + R\right] [I + K\Phi(z)B] =$$

$$R + \left[C_Q\Phi(z^{-1})B\right]^T \left[C_Q\Phi(z)B\right]$$

Kalman Filter & LQR Duality

$$[I + K\Phi(z)B]^T \left[B^T P B + R\right] \left[I + K\Phi(z^{-1})B\right] =$$

$$R + \left[C_Q \Phi(z) B\right]^T \left[C_Q \Phi(z^{-1}) B\right]$$

LQR	KF
P	M
A	A^T
B	C^T

LQR	KF
R	V
$C_Q{}^T$	$B'_w = B_w W^{1/2}$
K	L^T

$$\left[I + L^T \Phi^T(z) C^T\right]^T \left[C M C^T + V\right] \left[I + L^T \Phi^T(z^{-1}) C^T\right] =$$

$$V + \left[B_w'^T \Phi^T(z) C^T\right]^T \left[B_w'^T \Phi^T(z^{-1}) C^T\right]$$

KF return difference equality

From,

$$\left[I + L^T\Phi^T(z)C^T\right]^T \left[CMC^T + V\right] \left[I + L^T\Phi^T(z^{-1})C^T\right] =$$

$$V + \left[B_w'^T\Phi^T(z)C^T\right]^T \left[B_w'^T\Phi^T(z^{-1})C^T\right]$$

we perform transpose operations and notice that:

$$B_w' B_w'^T = B_w W B_w^T$$

This gives the desired result:

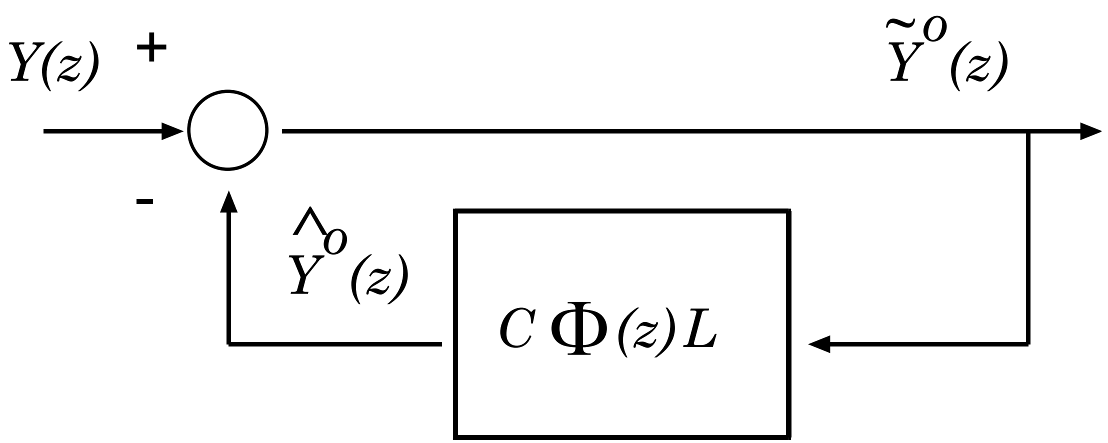
$$\boxed{\begin{aligned} &[I + C\Phi(z)L] \left[C\,MC^T + V\right] \left[I + C\Phi(z^{-1})L\right]^T = \\ &\qquad V + [C\Phi(z)B_w]\; W \left[C\Phi(z^{-1})B_w\right]^T \end{aligned}}$$

Kalman filter closed-loop eigenvalues

- A-priori KF (for $u(k) = 0$)

$$\begin{aligned} \widehat{x}^o(k+1) &= A\,\widehat{x}^o(k) + L\,\widetilde{y}^o(k) \\ \widehat{y}^o(k) &= C\widehat{x}^o(k) \end{aligned}$$

$$\widetilde{y}^o(k) = y(k) - C\,\widehat{x}^o(k)$$



Kalman filter closed-loop eigenvalues

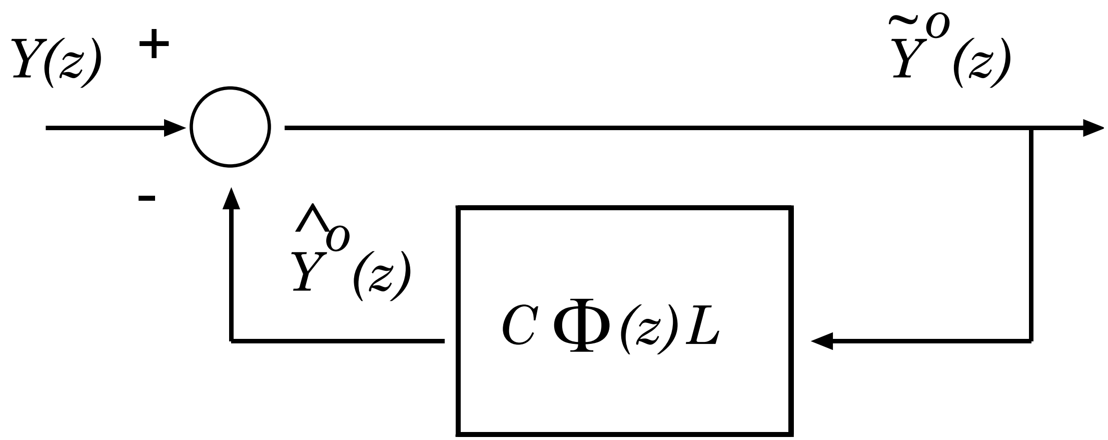
$$\widehat{x}^o(k+1) \;=\; \underbrace{(A - LC)}_{A_c}\,\widehat{x}^o(k) + L\,y(k)$$

•KF closed-loop eigenvalues

$$\widehat{C}(z) \;=\; \det\{(zI - A_c)\} = 0$$

$$\;=\; \det\{(zI - A + LC)\} = 0$$

Kalman filter return difference



$$\tilde{Y}^o(z) \;=\; [I + C\Phi(z)L]^{-1}\,Y(z)$$

Return difference: $[I + C\Phi(z)L]$

Kalman filter return difference

- Similar to the LQR case, we have that

$$\det\{[I + C\Phi(z)L]\} = \frac{\widehat{C}(z)}{\widehat{A}(z)}$$

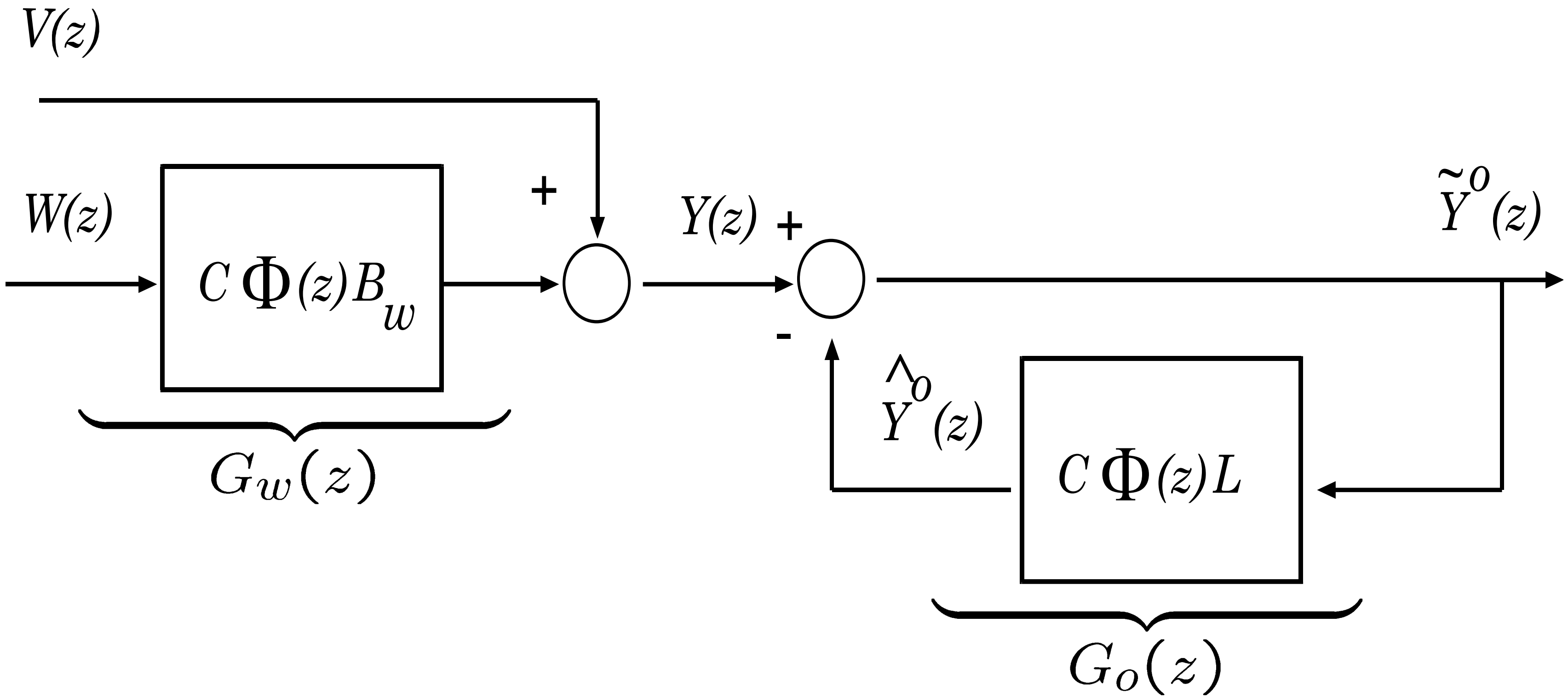
- KF closed-loop eigenvalues

$$\widehat{C}(z) = \det\{(zI - A + LC)\} = 0$$

- KF open-loop eigenvalues

$$\widehat{A}(z) = \det\{(zI - A)\} = 0$$

KF return difference equality

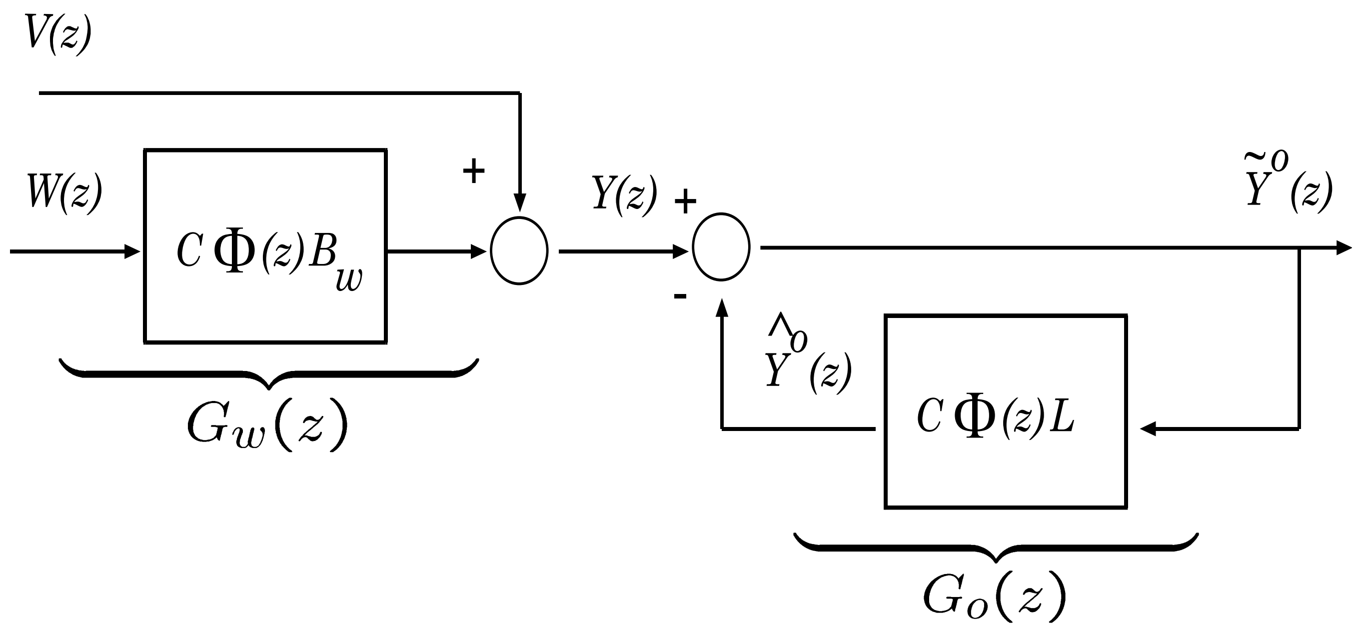


$$[I + G_o(z)]\,\left[C\,MC^T + V\right]\left[I + G_o(z^{-1})\right]^T =$$

$$V + G_w(z)\,W\,G_w^T(z^{-1})$$

KF return difference equality (SISO)

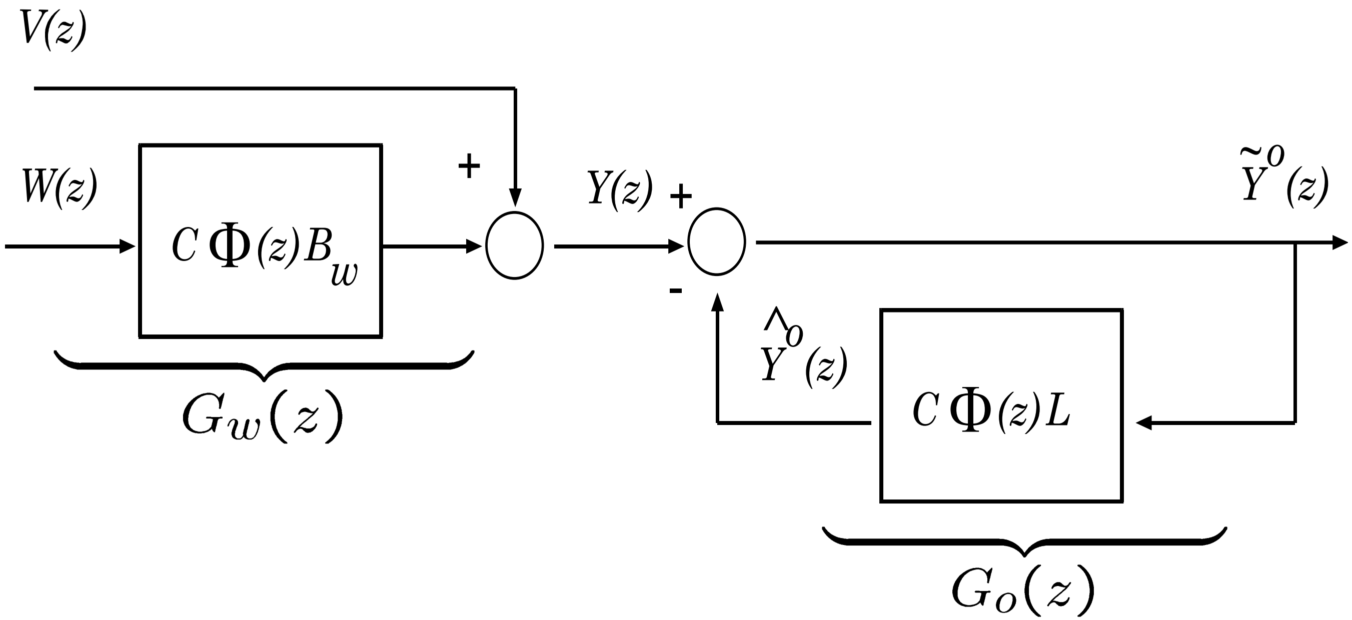
Assume that both, $w(k) \in \mathcal{R}$ and $y(k),\, v(k) \in \mathcal{R}$



$$\boxed{[1+G_o(z)][1+G_o(z^{-1})] = \gamma\,[1+\frac{W}{V}G_w(z)G_w(z^{-1})]}$$

$$\gamma = \frac{V}{V + C\,MC^T}$$

KF root locus for SISO Systems



$$[1 + G_o(z)] = [1 + C\Phi(z)L] = \frac{\widehat{C}(z)}{\widehat{A}(z)}$$

c.l. poles

o.l. poles

$$G_w(z) = C\Phi(z)B_w = \frac{\widehat{B}_w(z)}{\widehat{A}(z)}$$

o.l. zeros

o.l. poles

KF root locus for SISO Systems

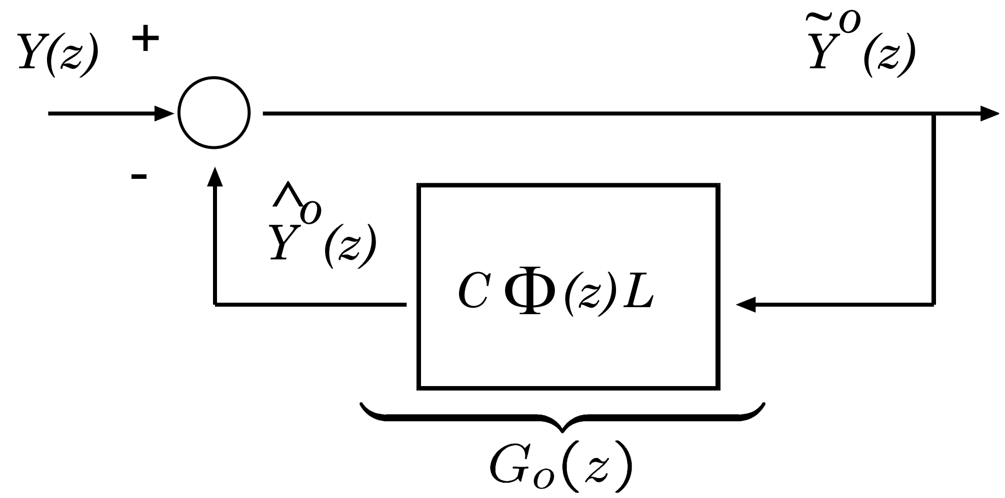
$$\boxed{\frac{\widehat{C}(z^{-1})\widehat{C}(z)}{\widehat{A}(z^{-1})\widehat{A}(z)} = \gamma\left[1+\rho\,\frac{\widehat{B}_w(z^{-1})\widehat{B}_w(z)}{\widehat{A}(z^{-1})\widehat{A}(z)}\right]}$$

$$\boxed{\rho = \frac{W}{V} \geq 0}$$

$$\frac{\textit{input noise intensity}}{\textit{measurement noise intensity}}$$

$$\gamma = \frac{V}{V + C\,MC^T} > 0, \quad \text{for} \quad V \in (0, \infty)$$

KF Loop phase margins (SISO)

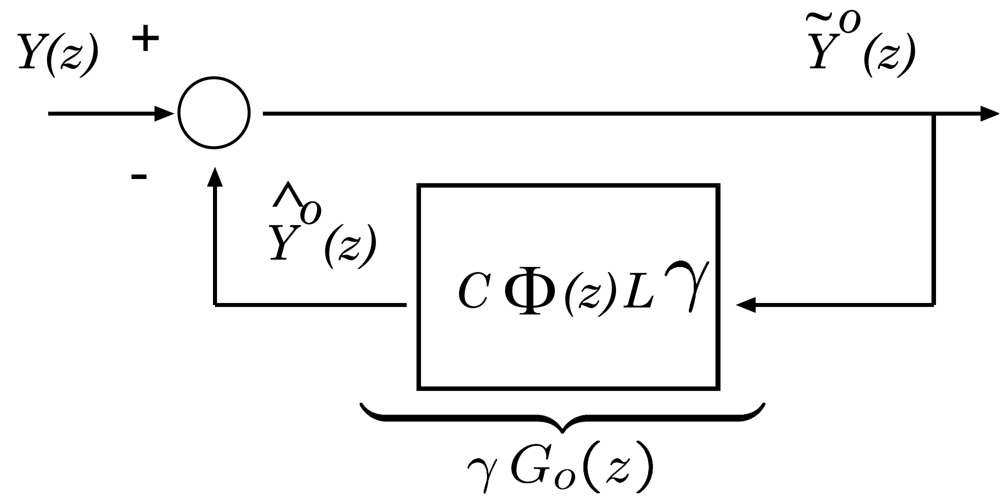


Utilizing LQR-KF duality, $$|(1+G_o(e^{j\omega}))| \geq \sqrt{\frac{V}{V+CMC^T}}$$

Therefore, a lower bound to the phase margin of $G_o(e^{j\omega})$ is:

$$\boxed{PM \geq 2\,\sin^{-1}\left\{0.5\sqrt{\frac{V}{V+CMC^T}}\right\}}$$

KF Loop gain margins (SISO)



Estimator was designed for $\gamma = 1$

Estimator is ***guaranteed*** to remain asymptotically stable for

$$\boxed{\frac{1}{1+\sqrt{V/(V+CMC^T)}} < \gamma < \frac{1}{1-\sqrt{V/V+CMC^T)}}}$$

Summary

- Stationary Kalman filters (KF):
  - KF algebraic Riccati equation
  - Convergence properties
- Kalman filter / LQR duality
- KF return difference equality
  - Reciprocal root locus
  - Guaranteed robustness margins