ME 233 Advanced Control II

Lecture 11
Kalman Filters Stationary Properties and LQR-KF Duality

(ME233 Class Notes pp.KF1-KF6)
Summary

• Stationary Kalman filters (KF):
  – KF algebraic Riccati equation
  – Convergence properties

• Kalman filter / LQR duality

• KF return difference equality
  – Reciprocal root locus
  – Guaranteed robustness margins
Stochastic State Estimation

Linear system contaminated by noise:

\[ Y = (zI - A)^{-1} CB + U \]

Two random disturbances:

- Input noise \( w(k) \) - contaminates the state \( x(k) \)
- Measurement noise \( v(k) \) - contaminates the output \( y(k) \)
Stochastic state model

State estimation of LTI system:

\[ x(k + 1) = Ax(k) + Bu(k) + Bw w(k) \]

\[ y(k) = Cx(k) + v(k) \]

Where:

- \( u(k) \) known control input
- \( w(k) \) Gaussian, uncorrelated, zero mean, input noise
- \( v(k) \) Gaussian, uncorrelated, zero mean, meas. noise
- \( x(0) \) Gaussian
Assumptions (review)

- Initial conditions:

\[
E\{x(0)\} = x_0 \quad E\{\tilde{x}^o(0)\tilde{x}^{oT}(0)\} = X_o
\]

- Noise properties:

\[
E\{w(k)\} = 0 \\
E\{v(k)\} = 0 \\
E\{w(k + l)w^T(k)\} = W(k) \delta(l) \\
E\{v(k + l)v^T(k)\} = V(k) \delta(l) \\
E\{w(k + l)v^T(k)\} = 0
\]

Zero-mean Gaussian uncorrelated noises

\[
E\{\tilde{x}^o(0)w^T(k)\} = 0 \quad E\{\tilde{x}^o(0)v^T(k)\} = 0
\]
Kalman Filter Solution V-1 (review)

A-posteriori state observer structure:

\[
\hat{x}(k) = \hat{x}^o(k) + F(k) \tilde{y}^o(k)
\]
\[
\hat{x}^o(k+1) = A \hat{x}(k) + B u(k)
\]
\[
\tilde{y}^o(k) = y(k) - C \hat{x}^o(k)
\]

\[
F(k) = M(k)C^T \left[ C M(k)C^T + V(k) \right]^{-1}
\]
\[
M(k+1) = AM(k)A^T + B_w W(k) B_w^T \\
- AM(k)C^T \left[ C M(k)C^T + V(k) \right]^{-1} CM(k)A^T
\]
Kalman Filter Solution V-1 (review)

- A-posteriori estimator as output
Kalman Filter Solution V-2 (review)

A-priori state observer structure:

\[
\begin{align*}
\hat{x}^o(k+1) &= A \hat{x}^o(k) + B u(k) + L(k) \tilde{y}^o(k) \\
\tilde{y}^o(k) &= y(k) - C \hat{x}^o(k)
\end{align*}
\]

\[
L(k) = A M(k) C^T \left[ C M(k) C^T + V(k) \right]^{-1}
\]

\[
M(k+1) = A M(k) A^T + B_w W(k) B_w^T - A M(k) C^T \left[ C M(k) C^T + V(k) \right]^{-1} C M(k) A^T
\]

\[
M(0) = X_o
\]
Kalman Filter Solution V-2 (review)

- Same structure as deterministic a-priori observer
Kalman Filter State Space (review)

\[ \hat{x}^o(k + 1) = [A - L(k)C]\hat{x}^o(k) + [B \quad L(k)]\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \]

\[ \hat{x}(k) = [I - F(k)C]\hat{x}^o(k) + [0 \quad F(k)]\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \]

\[ F(k) = M(k)C^T \left[ C M(k)C^T + V(k) \right]^{-1} \]

\[ L(k) = A M(k)C^T \left[ C M(k)C^T + V(k) \right]^{-1} \]

\[ M(k + 1) = A M(k)A^T + B_w W(k) B_w^T \]

\[ - A M(k)C^T \left[ C M(k)C^T + V(k) \right]^{-1} C M(k)A^T \]
Kalman Filter (KF) Properties (review)

The KF a-priori output error \( (a\text{-priori output residual}) \)

\[
\tilde{y}^o(k) = y(k) - C \tilde{x}^o(k)
\]

is often called the **innovation**

it contains only the “new information” in \( y(k) \)

Moreover,

\[
\Lambda_{\tilde{y}^o\tilde{y}^o}(k, j) = [CM(k)C^T + V(k)]\delta(j)
\]

i.e. \( \tilde{y}^o(k) \) is an uncorrelated RVS
For the figure on the next slide, we will assume without loss of generality that the control input is zero, i.e.

\[ u(k) = 0 \quad k = 0, 1, \ldots \]

- **Plant:**
  \[ x(k + 1) = A x(k) + B_w w(k) \]
  \[ y(k) = C x(k) + v(k) \]

- **Kalman filter V-2:**
  \[ \hat{x}^o(k + 1) = A \hat{x}^o(k) + L(k) \tilde{y}^o(k) \]
  \[ \hat{y}^o(k) = C \hat{x}^o(k) \]
KF as an innovations filter (review)

\[ \Phi(z) = (zI - A)^{-1} \]

\[
\begin{bmatrix}
V(k) & 0 \\
0 & W(k)
\end{bmatrix}
\]

Uncorrelated noise input

Correlated noise output

\[
CM(k)C^T + V(k)
\]

Uncorrelated noise output
Kalman Filter (KF) Properties (review)

• The KF is a linear time varying estimator.
• The KF is the **optimal state estimator** when the input and measurement noises are Gaussian.
• The KF is still the **optimal linear state estimator** even when the input and measurement noises are **not** Gaussian.
• The KF covariance Riccati equation is iterated in a forward manner, rather than in a backwards manner as in the LQR.

\[ M(0) \rightarrow M(k) \]
Steady State Kalman Filter

• Assume now that we want to estimate the state under zero-mean, stationary input and output Gaussian white noise, i.e.

\[ x(k + 1) = Ax(k) + Bu(k) + B_ww(k) \]
\[ y(k) = Cx(k) + v(k) \]

\[
\begin{align*}
E\{w(k)\} &= 0 \\
E\{v(k)\} &= 0 \\
E\{w(k + l)w^T(k)\} &= W \delta(l) \\
E\{v(k + l)v^T(k)\} &= V \delta(l) \\
E\{w(k + l)v^T(k)\} &= 0
\end{align*}
\]

WSS Gaussian Noise
A priori estimation error dynamics

\[ \ddot{x}^o(k+1) = [A - L(k)C] \ddot{x}^o(k) + B_w w(k) - L(k)v(k) \]

Proof:

\[
\begin{align*}
    x(k + 1) &= Ax(k) + Bu(k) + B_w w(k) \\
    \ddot{x}^o(k + 1) &= A\ddot{x}^o(k) + Bu(k) + L(k)\ddot{y}^o(k)
\end{align*}
\]

Subtracting equations gives

\[
\ddot{x}^o(k + 1) = A\ddot{x}^o(k) + B_w w(k) - L(k)\ddot{y}^o(k) \]

\[ C\ddot{x}^o(k) + v(k) \]
Steady state Kalman filter, question 1

1) When does there exist a **BOUNDDED limiting** solution 

\[ M_\infty \]

for each choice of \( M(0) \geq 0 \)?

\[
M(k + 1) = AM(k)A^T + B_w W B_w^T \\
- AM(k)C^T [CM(k)C^T + V]^{-1} CM(k)A^T
\]
Steady state Kalman filter, question 2

2) When does there exist a **UNIQUE limiting** solution 

\[ M_\infty \]

to the Riccati Eq.

\[
M(k + 1) = AM(k)A^T + B_wWB_w^T - AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T
\]

**regardless** of the choice of \( M(0) \succeq 0 \) ?
Steady state Kalman filter, question 3

3) When does the limiting solution

\[ M_\infty \]

to the Riccati Eq.

yield asymptotically stable estimation error dynamics?

\[ A_c = A - L_\infty C \]

is Schur
(all eigenvalues inside unit circle)

\[ L_\infty = A M_\infty C^T \left[ C M_\infty C^T + V \right]^{-1} \]
Detectability Assumption

We are only interested in the case where the estimation error dynamics are asymptotically stable.

If \((C,A)\) is not detectable, then there does not exist a estimator that results in asymptotically stable estimation error dynamics.

→ For the stationary Kalman filter, we always assume that \((C,A)\) is detectable.
Theorem 1 : Existence of a bounded $M_\infty$

Let $(C, A)$ be detectable
(unobservable modes are asymptotically stable)

Then, for $M(0) = X_0 = 0$ as $k \to \infty$
the solution of the Riccati Eq.

$$M(k+1) = AM(k)A^T + B_wWB_w^T$$
$$- AM(k)C^T[CM(k)C^T + V]^{-1}CM(k)A^T$$

converges to a **bounded limiting** solution $M_\infty$
that satisfies the algebraic Riccati equation (DARE):

$$M_\infty = AM_\infty A^T + B_wWB_w^T$$
$$- AM_\infty C^T[CM_\infty C^T + V]^{-1}CM_\infty A^T$$
Theorem 1: Notes

- Theorem 1 only guarantees the existence of a bounded solution $M_\infty$ to the algebraic Riccati Equation

\[ M_\infty = AM_\infty A^T + B_w W B_w^T - AM_\infty C^T [CM_\infty C^T + V]^{-1} C M_\infty A^T \]

- The solution may not be unique.

- Different initial conditions $M(0) = X_0$ may result in different limiting solutions $M_\infty$ or may yield no limiting solution at all!
Theorem 2: Existence and uniqueness of a positive definite asymptotic stabilizing solution

If \((C,A)\) is detectable and \((A,B_wW^{1/2})\) is controllable

1) There exists a unique, bounded solution \(M_\infty \succ 0\) to the DARE

\[
M_\infty = AM_\infty A^T + B_wW B_w^T - AM_\infty C^T [CM_\infty C^T + V]^{-1} CM_\infty A^T
\]

2) The estimation error dynamics are asymptotically stable

\[
\ddot{x}^o(k + 1) = [A - L_\infty C] \ddot{x}^o(k) + B_w w(k) - L_\infty v(k)
\]

\[
L_\infty = AM_\infty C^T [CM_\infty C^T + V]^{-1}
\]
Theorem 3: Existence of a stabilizing solution

If $(C,A)$ is detectable and $(A,B_wW^{1/2})$ is stabilizable

1) There exists a unique, bounded solution $M_\infty \succeq 0$ to the DARE

\[
M_\infty = AM_\infty A^T + B_wW B_w^T - AM_\infty C^T [CM_\infty C^T + V]^{-1} CM_\infty A^T
\]

2) The estimation error dynamics are \textbf{asymptotically stable}

\[
\ddot{x}^o(k + 1) = [A - L_\infty C] \ddot{x}^o(k) + B_ww(k) - L_\infty v(k)
\]

\[
L_\infty = AM_\infty C^T [CM_\infty C^T + V]^{-1}
\]
Theorem 4: A different approach

The discrete algebraic Riccati equation (DARE) has a solution for which $A - L_\infty C$ is Schur

if and only if

$(C, A)$ is detectable and the matrix pair $(A, B_w W^{1/2})$ has no uncontrollable modes on the unit circle.

\[
L_\infty = AM_\infty C^T [CM_\infty C^T + V]^{-1}
\]

\[
M_\infty = AM_\infty A^T + B_w W B_w^T
\]

\[
-AM_\infty C^T [CM_\infty C^T + V]^{-1} CM_\infty A^T
\]
Kalman Filter Solution V-1

A-posteriori state observer structure:

\[ \hat{x}(k) = \hat{x}^o(k) + F \tilde{y}^o(k) \]
\[ \hat{x}^o(k + 1) = A \hat{x}(k) + B u(k) \]
\[ \tilde{y}^o(k) = y(k) - C \hat{x}^o(k) \]

\[
F = M C^T \left[ C M C^T + V \right]^{-1}
\]
\[
M = A M A^T + B_w W B_w^T - A M C^T \left( C M C^T + V \right)^{-1} C M A^T
\]

\[ A - AFC \text{ is Schur} \]
Kalman Filter Solution V-2

A-priori state observer structure:

\[ \hat{x}^o(k + 1) = A\hat{x}^o(k) + Bu(k) + L\tilde{y}^o(k) \]
\[ \tilde{y}^o(k) = y(k) - C\hat{x}^o(k) \]

\[ L = AMCT \left[ CMC^T + V \right]^{-1} \]
\[ M = AMA^T + BwWB_w^T - AMCT( CMC^T + V )^{-1} CMA^T \]
\[ A - LC \text{ is Schur} \]
Kalman Filter State Space

\[
\hat{x}_o(k + 1) = [A - LC]\hat{x}_o(k) + [B \quad L] \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}
\]

\[
\hat{x}(k) = [I - FC]\hat{x}_o(k) + [0 \quad F] \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}
\]

\[
F = M C^T \left[ C M C^T + V \right]^{-1}
\]

\[
L = A M C^T \left[ C M C^T + V \right]^{-1}
\]

\[
M = A M A^T + B_w W B_w^T - A M C^T (C M C^T + V)^{-1} C M A^T
\]

\[A - LC \text{ is Schur}\]
Kalman Filter (KF) Properties

The KF a-priori output error (*a-priori output residual*)

\[ \tilde{y}^o(k) = y(k) - C \hat{x}^o(k) \]

is often called the **innovation**

it contains only the “new information” in \( y(k) \)

Moreover,

\[ \Lambda_{y^o y^o}(j) = [CMC^T + V] \delta(j) \]

i.e. \( \tilde{y}^o(k) \) is white
KF as an innovations filter

For the figure on the next slide, we will assume without loss of generality that the control input is zero, i.e.

\[ u(k) = 0 \quad k = 0, 1, \ldots \]

- **Plant:**
  \[ x(k+1) = A x(k) + B_w w(k) \]
  \[ y(k) = C x(k) + v(k) \]

- **Kalman filter V-2:**
  \[ \hat{x}^o(k+1) = A\hat{x}^o(k) + L\hat{y}^o(k) \]
  \[ \hat{y}^o(k) = C\hat{x}^o(k) \]
KF as an innovations (whitening) filter

\[ \Phi(z) = (zI - A)^{-1} \]

**Diagram:**
- **Plant:**
  - Input: \( V(z) \)
  - Input: \( W(z) \)
  - Output: \( C \Phi(z)B_w \)

- **Kalman Filter:**
  - Input: \( Y(z) \)
  - Output: \( \tilde{Y}^0(z) \)
  - Gain: \( C \Phi(z)L \)

**Noise:**
- White noise input: \( \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix} \)
- Colored noise output: \( \begin{bmatrix} \Lambda^0 \tilde{Y}(z) \\ C \Phi(z)L \end{bmatrix} \)
- White noise output: \( CMCT^T + V \)
Output $Y(k)$ is colored noise

- **Plant:**
  \[
  x(k + 1) = A x(k) + B_w w(k) \\
  y(k) = C x(k) + v(k)
  \]

\[
\Phi(z) = (zI - A)^{-1}
\]
Output $Y(k)$ is colored noise

- Plant:

$$Y(z) = [C \Phi(z) B_w] W(z) + V(z)$$

$$\Phi(z) = (zI - A)^{-1}$$
KF as an innovations filter

- A-priori KF:

\[
\hat{x}^o(k+1) = A \hat{x}^o(k) + L \tilde{y}^o(k)
\]

\[
\tilde{y}^o(k) = y(k) - C \hat{x}^o(k)
\]

\[
\Phi(z) = (zI - A)^{-1}
\]
KF as an innovations filter

- **A-priori KF:**

\[
\tilde{Y}^o(z) = [I + C\Phi(z)L]^{-1} Y(z)
\]

\[
\Phi(z) = (zI - A)^{-1}
\]
KF as an innovations filter

- Plant
  \[ Y(z) = [C \Phi(z) B_w] W(z) + V(z) \]

- A-priori KF:
  \[ \tilde{Y}^o(z) = [I + C \Phi(z) L]^{-1} Y(z) \]
$Y(k)$ Power spectrum

Power spectrum of $y(k)$

\[
Y(z) = [C\Phi(z)B_w] W(z) + V(z)
\]

$Y_w(z)$

\[w(k)\text{ and } v(k)\text{ are uncorrelated!}\]
$Y(k)$ Power spectrum

Power spectrum of $y(k)$

$$Y(z) = [C \Phi(z) B_w] W(z) + V(z)$$

$Y_w(z)$

$y_w(k)$ and $v(k)$ are also uncorrelated!
The power spectrum $Y(k)$ of $y(k)$ is given by:

$$Y(z) = [C \Phi(z) B_w] W(z) + V(z)$$

Where $W(z)$ and $V(z)$ are the power spectra of $W(z)$ and $V(z)$, respectively.

The diagram represents the relationship between $V(z)$, $W(z)$, and $Y(z)$ as:

- $V(z)$ is an input to the system.
- $W(z)$ is an input to the system.
- The output of the system is $Y(z)$.
- The block labeled $C \Phi(z) B_w$ represents the transfer function of the system.

The equation for the power spectrum is:

$$\Lambda_{yy}(z) = \Lambda_{yw} y_w(z) + \Lambda_{vv}(z)$$
$Y(k)$ Power spectrum

Power spectrum of $y(k)$

\[ V(z) \]

\[ W(z) \]

\[ C \Phi(z) B_w \]

\[ Y_w(z) + \]

\[ Y(z) \]

\[ \Lambda_{yy}(z) = \Lambda_{yw y_w}(z) + \Lambda_{vv}(z) \]

$\nu(k)$ is white noise
\[ Y(k) \text{ Power spectrum} \]

\[ w(k) \rightarrow G(k) \rightarrow y_w(k) \]

\[ \Lambda_{y_w y_w}(z) = G(z) \Lambda_{w_w}(z) G^T(z^{-1}) \]
$Y(k)$ Power spectrum

Power spectrum of $y(k)$

$V(z)$

$W(z) \quad G(z) \quad Y_w(z) + \quad Y(z)$

$\Lambda_{yy}(z) = \Lambda_{yw}y_w(z) + V$

$\Lambda_{yw}y_w(z) = [C\Phi(z)B_w] W [C\Phi(z^{-1})B_w]^T$

$w(k)$ is white noise
$Y(k)$ Power spectrum

Power spectrum of $y(k)$

$V(z)$

$W(z) \rightarrow C \Phi(z) B_w \rightarrow Y_w(z) + \rightarrow Y(z)$

$\Lambda_{yy}(z) = V + [C\Phi(z)B_w] W \left[ C\Phi(z^{-1})B_w \right]^T$
KF as an innovations filter

Power spectrum of $\tilde{y}^o(k)$

\[ \tilde{Y}^o(z) = [I + C\Phi(z)L]^{-1} Y(z) \]

\[ \Lambda_{\tilde{y}o\tilde{y}o}(z) = [I + C\Phi(z)L]^{-1} \Lambda_{yy}(z) \left[ I + C\Phi(z^{-1})L \right]^{-T} \]

\[ \Lambda_{yy}(z) = [I + C\Phi(z)L] \Lambda_{\tilde{y}o\tilde{y}o}(z) \left[ I + C\Phi(z^{-1})L \right]^T \]
**KF as an innovations filter**

Combining two results:

\[ V(z) \]

\[ W(z) \]

\[ C \Phi(z) B_w \]

\[ + \]

\[ Y(z) \]

\[ \Lambda_{yy}(z) = V + [C \Phi(z) B_w] \ W \ [C \Phi(z^{-1}) B_w]^T \]

and

\[ \Lambda_{yy}(z) = [I + C \Phi(z)L] \ \Lambda_{\tilde{y}o\tilde{y}o}(z) \ [I + C \Phi(z^{-1})L]^T \]
KF as an innovations filter

Combining two results:

\[
V(z) \rightarrow W(z) \rightarrow C \Phi(z) B_w \rightarrow + \rightarrow Y(z) \rightarrow + \rightarrow \tilde{Y}^o(z) \rightarrow C \Phi(z) L
\]

\[
[I + C\Phi(z)L] \Lambda_{\tilde{y}^o\tilde{y}^o}(z) \left[ I + C\Phi(z^{-1})L \right]^T = V + [C\Phi(z)B_w] W [C\Phi(z^{-1})B_w]^T
\]
KF as an innovations filter

Recall what Theorem part 3) says about the a-priori output error (the innovation sequence)

\[ \Lambda_{\tilde{y}^o \tilde{y}^o}(l) = E \{ \tilde{y}^o(k + l)\tilde{y}^{oT}(k) \} \]

\[ = \left[ C \ M C^T + V \right] \delta(l) \]

\[ \tilde{y}^o(k) \] is also white noise!!
KF as an innovations filter

Recall what Theorem part 3) says about the a-priori output error (the innovation sequence)

\[ \Lambda_{\tilde{y}^o\tilde{y}^o}(l) = [CMC^T + V] \delta(l) \]

\[ \Lambda_{\tilde{y}^o\tilde{y}^o}(z) = [CMC^T + V] \]

\[ \Phi_{\tilde{y}^o\tilde{y}^o}(\omega) = [CMC^T + V] \]
KF as a innovations (whitening) filter

\[ W(z) \]

\[ Y(z) + C B w \Phi(z) - \]

\[ L Y(z) o Y(z) o^\sim C \Phi(z) \]

plant

Kalman filter

\[ \tilde{Y}^0(z) \]

whitening filter
KF as an innovations filter

\[ \Lambda_{\tilde{y}_o \tilde{y}_o}(z) = \left[ C M C^T + V \right] \]

Therefore,

\[ [I + C\Phi(z)L] \left[ C M C^T + V \right] \left[ I + C\Phi(z^{-1})L \right]^T = \Lambda_{\tilde{y}_o \tilde{y}_o}(z) \]

\[ V + \left[ C\Phi(z)B_w \right] W \left[ C\Phi(z^{-1})B_w \right]^T = \Lambda_{yy}(z) \]
KF return difference equality

\[
[I + C\Phi(z)L] \left[ CMC^T + V \right] [I + C\Phi(z^{-1})L]^T = \]

\[
V + [C\Phi(z)B_w] W [C\Phi(z^{-1})B_w]^T
\]
Kalman Filter & LQR Duality

Recall Steady state LQR:

\[ x(k + 1) = A x(k) + B u(k) \]
\[ u(k) = -K x(k) + r(k) \]

\[ J = \sum_{k=0}^{\infty} \left\{ x^T(k) C_Q^T C_Q x(k) + u^T(k) R u(k) \right\} \]

\[ Q = C_Q^T C_Q \geq 0 \]
\[ R = R^T > 0 \]
Note:
We need to distinguish between:

- **LQR**: state cost weight \( Q = C_Q^T C_Q \geq 0 \)

\[
J = \sum_{k=0}^{\infty} \left\{ x^T(k) C_Q^T C_Q x(k) + u^T(k) R u(k) \right\}
\]

- **KF**: output matrix \( C \)

\[
x(k + 1) = Ax(k) + Bu(k) + Bw w(k)
\]

\[
y(k) = C x(k) + v(k)
\]
Kalman Filter & LQR Duality

Infinite-horizon LQR  Closed-loop dynamics:

\[ x(k+1) = (A - BK)x(k) + Br(k) \]

\[ K = \left[ R + B^T PB \right]^{-1} B^T PA \]

\[ A^T PA - P = -C_Q^T C_Q \]

\[ + A^T PB \left[ B^T PB + R \right]^{-1} B^T PA \]
Kalman Filter & LQR Duality

Steady State KF Estimation error dynamics

\[ \tilde{x}^o(k + 1) = (A - LC) \tilde{x}^o(k) + B_w w(k) - L v(k) \]

\[ L = A M C^T \left[ C M C^T + V \right]^{-1} \]

\[ A M A^T - M = -B_w W B_w^T \]

\[ + A M C^T \left[ C M C^T + V \right]^{-1} C M A^T \]
Kalman Filter & LQR Duality

Let’s compare the DAREs:

\[
A^T P A - P = -C_t^T C \tag{LQR}
\]

\[
+ A^T P B \left[ B^T P B + R \right]^{-1} B^T P A
\]

\[
A M A^T - M = -B_w W B_w^T \tag{KF}
\]

\[
+ A M C^T \left[ C M C^T + V \right]^{-1} C M A^T
\]

\[P \Rightarrow M\]
Kalman Filter & LQR Duality

Let’s compare the AREs:

\[ A^T P A - P = -C_Q^T C_Q \]

\[ + A^T P B \left[ B^T P B + R \right]^{-1} B^T P A \]

\[ A M A^T - M = -B_w W B_w^T \]

\[ + A M C^T \left[ C M C^T + V \right]^{-1} C M A^T \]

\[ C_Q^T \Rightarrow B_w W^{1/2} = B_w' \]
Kalman Filter & LQR Duality

Let’s compare the AREs:

\[ A^T P A - P = -C_Q^T C_Q \]
\[ + A^T P B \left[ B^T P B + R \right]^{-1} B^T P A \]

\[ AM A^T - M = -B_w W B_w^T \]
\[ + A M C^T \left[ C M C^T + V \right]^{-1} C M A^T \]

\[ A \Rightarrow A^T \]
Kalman Filter & LQR Duality

Let’s compare the AREs:

\[ A^T P A - P = -C_Q^T C_Q \]

\[ + A^T P B \left[ B^T P B + R \right]^{-1} B^T P A \]

\[ A M A^T - M = -B_w W B_w^T \]

\[ + A M C^T \left[ C M C^T + V \right]^{-1} C M A^T \]

\[ B \Rightarrow C^T \]
Kalman Filter & LQR Duality

Let's compare the AREs:

\[ A^T P A - P = -C_Q^T C_Q \]

\[ + A^T P B \left[ B^T P B + R \right]^{-1} B^T P A \]

\[ A M A^T - M = -B_w W B_w^T \]

\[ + A M C^T \left[ C M C^T + V \right]^{-1} C M A^T \]

\[ R \Rightarrow V \]
Kalman Filter & LQR Duality

Let’s compare the Feedback gains:

\[ K = \left[ R + B^T P B \right]^{-1} B^T P A \]

\[ L^T = \left[ V + C M C^T \right]^{-1} C M A^T \]

\[ P \Rightarrow M \quad B \Rightarrow C^T \quad A \Rightarrow A^T \quad R \Rightarrow V \]
Kalman Filter & LQR Duality

Let’s compare the Feedback gains:

\[ K^T = APB \left[ R + B^TPB \right]^{-1} \]

\[ L = AMC^T \left[ V + CMC^T \right]^{-1} \]

\[ K^T \Rightarrow L \]
Kalman Filter & LQR Duality

Comparing ARE’s and feedback gains, we obtain the following duality

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## Kalman Filter & LQR Duality

### Duality Table:

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### Equations:

1. \[ A^T PA - P + C_Q^T C_Q - A^T PB \left[ B^T PB + R \right]^{-1} B^T PA = 0 \]

2. \[ A M A^T - M + B'_w B'_w - A M C^T \left[ C M C^T + V \right]^{-1} C M A^T = 0 \]
Kalman Filter & LQR Duality

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\[ K = \left( B^T P B + R \right)^{-1} \quad B^T P A \]

\[ L^T = \left[ C M C^T + V \right]^{-1} \quad C M A^T \]
Kalman Filter & LQR Duality

• It is possible to use duality to prove theorems 1-4 for stationary Kalman filters from the corresponding theorems from the infinite horizon LQR

• The following slides give an outline of how to do this

• The main idea is to design an infinite horizon LQR for a fictitious system
Theorems 1-4 proof methodology

- Consider the LQR problem:

\[ \bar{x}(k + 1) = A^T \bar{x}(k) + C^T \bar{u}(k) \]

\[ J = \bar{x}^T(N) X_0 \bar{x}(N) + \sum_{k=0}^{N-1} \left\{ \bar{x}^T(k) B_w W B_w^T \bar{x}(k) + \bar{u}^T(k) V \bar{u}(k) \right\} \]

- Solution:

\[ \bar{u}(k) = -[C \bar{P}(k + 1) C^T + V]^{-1} C \bar{P}(k + 1) A^T \bar{x}(k) \]

\[ \bar{P}(k - 1) = A \bar{P}(k) A^T + B_w W B_w^T \]
\[- A \bar{P}(k) C^T [C \bar{P}(k) C^T + V]^{-1} C \bar{P}(k) A^T \]

\[ \bar{P}(N) = X_0 = M(0) \]
Theorems 1-4 proof methodology

• The solution of the Riccati equation

\[
\bar{P}(k - 1) = A\bar{P}(k)A^T + B_wW B_w^T
- A\bar{P}(k)C^T [C\bar{P}(k)C^T + V]^{-1}C\bar{P}(k)A^T
\]

\[
\bar{P}(N) = X_0 = M(0)
\]

is

\[
\bar{P}(N - k) = M(k)
\]

• Use LQR convergence results for \( \bar{P}(0) \) as \( N \to \infty \) to yield convergence results for \( \tilde{M}(N) \) as \( N \to \infty \)
Theorems 1-4 proof methodology

- Other key ideas in proofs
  
  - \((A^T, C^T)\) stabilizable iff \((C, A)\) detectable
  
  - Unobservable modes of \(((B_w W^{1/2})^T, A^T)\) are the uncontrollable modes of \((A, B_w W^{1/2})\)
  
  - \(A^T C^T L^T\) is Schur iff \(A-LC\) is Schur
Steady State LQR

Theorem 1):

If the pair \([A, B]\) is controllable (or stabilizable), the solution of the DRE

\[-P(k) = A^T P(k + 1) A + C_Q^T C_Q - A^T P(k + 1) B \left[ B^T P(k + 1) B + R \right]^{-1} B^T P(k + 1) A\]

converges, as \(N \to \infty\), to a constant that satisfies

\[P = A^T P A + C_Q^T C_Q - A^T P B \left[ B^T P B + R \right]^{-1} B^T P A\]
Steady State LQR

Theorem 2:
If the pair \([A, C_q]\) is observable (or detectable)

Then \([A, B]\) is controllable (or stabilizable) if and only if:

1) The solution of

\[-P(k) = A^T P(k + 1) A + C_Q^T C_Q\]

\[- A^T P(k + 1) B \left[ B^T P(k + 1) B + R \right]^{-1} B^T P(k + 1) A\]

with \(P(N) \succeq 0\)

Converges to a **unique** stationary solution \(P\), which satisfies

\[P = A^T P A + C_Q^T C_Q - A^T P B \left[ B^T P B + R \right]^{-1} B^T P A\]
Steady State LQ

Theorem 2: (continuation)

2) $P$ is positive definite (semi-definite)

3) The close loop matrix $A_c = A - BK$ is Schur

$$K = \left[ BT PB + R \right]^{-1} B^T PA$$
Kalman Filter & LQR Duality

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$LQR$\[ [A, B] \text{ controllable} \quad \xrightarrow{duality} \quad [A^T, C^T] \text{ controllable} \]

$KF$\[ [C, A] \text{ observable} \]
# Kalman Filter & LQR Duality

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**LQR**

$[C_Q, A]$ observable

**duality**

**KF**

$[B'_w, A^T]$ observable

$[A, B'_w]$ controllable
Steady State Kalman Filter

Theorem 1:
If the pair $[A, C]$ is observable (or detectable):
the solution of

$$M(k+1) = AM(k)A^T + B_wWB_w^T$$

$$- AM(k)C^T [CM(k)C^T + V]^{-1} CM(k)A^T$$

with $M(0) = 0$

Converges to a stationary solution, $M$, which satisfies

$$M = AMA^T + B_wWB_w^T - AMCT [CMCT + V]^{-1} CMAT$$
Steady State Kalman Filter

Theorem 2:

If the pair $[A, B'_w]$ is controllable (or stabilizable), where

$$B'_w = B_w W^{1/2}$$

Then $[A, C]$ is observable (or detectable) if and only if:

1) The solution of

$$M(k + 1) = AM(k)A^T + B_w W B'^T_w$$

$$- AM(k)C^T [CM(k)C^T + V]^{-1} CM(k)A^T$$

Converges to a unique stationary solution $M$, which satisfies

$$M = AM A^T + B_w W B'^T_w - AM C^T [CM C^T + V]^{-1} CM A^T$$
Steady State Kalman Filter

Theorem 2: (continuation)

2) $M$ is positive definite (semi-definite)

3) The close loop matrix $A_c = A - LC$
   is Schur

   $$L = AMCT \left[ CMC^T + V \right]^{-1}$$
Steady State Kalman Filter

Theorem 3:
Under stationary noise and the conditions in theorems 1) and 2),

The observer a-priori residual (innovations)

\[ \tilde{y}^o(k) = y(k) - C \hat{x}^o(k) \]

is white

\[ E \left\{ \tilde{y}^o(k + l)\tilde{y}^{oT}(k) \right\} = \left[ CMC^T + V \right] \delta(l) \]
KF as an innovations filter

We will assume, without loss of generality that the control input is zero, i.e.

\[ u(k) = 0 \quad k = 0, 1, \ldots \]

- **Plant:**

\[
\begin{align*}
    x(k + 1) &= A x(k) + B_w w(k) \\
    y(k) &= C x(k) + v(k)
\end{align*}
\]
KF as an innovations (whitening) filter

\[ W(z) Y(z) + C B w \Phi(z) - L Y(z) o \sim C \Phi(z) \]

White noise input \[ [W + V] \]

Colored noise output

White noise output \[ [C MC^T + V] \]
KF as a innovations (whitening) filter

\[ W(z) \]

\[ C \Phi(z) B_w \]

\[ Y(z) \]

\[ \bar{Y}^0(z) \]

\[ 
\begin{align*}
C \Phi(z) L \\
\bar{Y}^0(z)
\end{align*}
\]

White noise input

\([W + V]\)

Colored noise output

White noise output

\([CMC^T + V]\)
Return difference equality for LQR (review)

Open loop transfer function:

\[ G_o(z) = K \Phi(z)B \]

TF from \( U(z) \) to \( Y_Q(z) \):

\[ G_Q(z) = C_Q \Phi(z)B \]
Return difference equality for LQR (review)

Substituting, \[ G_o(z) = K \Phi(z)B \quad G_Q(z) = C_Q \Phi(z)B \]

into

\[ [I + G_o(z^{-1})]^T [R + B^T P B] [I + G_o(z)] = R + G_Q^T(z^{-1}) G_Q(z) \]

We obtain,

\[ [I + K \Phi(z^{-1})B]^T [B^T P B + R] [I + K \Phi(z)B] = \]

\[ R + [C_Q \Phi(z^{-1})B]^T [C_Q \Phi(z)B] \]
Kalman Filter & LQR Duality

\[
[I + K \Phi(z) B]^T [B^T P B + R] [I + K \Phi(z^{-1}) B] = \\
R + \left[ C_Q \Phi(z) B \right]^T \left[ C_Q \Phi(z^{-1}) B \right]
\]

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\[
[I + L^T \Phi^T(z) C^T]^T \left[ C M C^T + V \right] [I + L^T \Phi^T(z^{-1}) C^T] = \\
V + \left[ B'_w^T \Phi^T(z) C^T \right]^T \left[ B'_w^T \Phi^T(z^{-1}) C^T \right]
\]
KF return difference equality

From,

\[
\left[I + L^T \Phi^T(z)C^T\right]^T \left[C MC^T + V\right] \left[I + L^T \Phi^T(z^{-1})C^T\right] =
\]

\[
V + \left[B'_w \Phi^T(z)C^T\right]^T \left[B'_w \Phi^T(z^{-1})C^T\right]
\]

we perform transpose operations and notice that:

\[
B'_w B'_w = B_w W B'_w
\]

This gives the desired result:

\[
\left[I + C \Phi(z) L\right] \left[C MC^T + V\right] \left[I + C \Phi(z^{-1}) L\right]^T =
\]

\[
V + [C \Phi(z) B_w] W \left[C \Phi(z^{-1}) B_w\right]^T
\]
Kalman filter closed-loop eigenvalues

• A-priori KF (for $u(k) = 0$)

$$\hat{x}^o(k + 1) = A \hat{x}^o(k) + L \tilde{y}^o(k)$$

$$\tilde{y}^o(k) = C \hat{x}^o(k)$$

$$\tilde{y}^o(k) = y(k) - C \hat{x}^o(k)$$
Kalman filter closed-loop eigenvalues

\[
\hat{x}^o(k + 1) = \underbrace{(A - LC)}_{A_c} \hat{x}^o(k) + L y(k)
\]

- KF closed-loop eigenvalues

\[
\hat{C}(z) = \det\{(zI - A_c)\} = 0
\]

\[
= \det\{(zI - A + LC)\} = 0
\]
Kalman filter return difference

\[ \tilde{Y}^o(z) = [I + C\Phi(z)L]^{-1} Y(z) \]

Return difference: \[ [I + C\Phi(z)L] \]
Kalman filter return difference

• Similar to the LQR case, we have that

\[
\text{det}\{[I + C\Phi(z)L]\} = \frac{\hat{C}(z)}{\hat{A}(z)}
\]

• KF closed-loop eigenvalues

\[
\hat{C}(z) = \text{det}\{(zI - A + LC)\} = 0
\]

• KF open-loop eigenvalues

\[
\hat{A}(z) = \text{det}\{(zI - A)\} = 0
\]
KF return difference equality

\[ V(z) \]

\[ W(z) \]

\[ C \Phi(z) B_w \]

\[ G_w(z) \]

\[ Y(z) + \]

\[ \tilde{Y}^0(z) \]

\[ Y(z) - \]

\[ Y^0(z) \]

\[ C \Phi(z) L \]

\[ G_o(z) \]

\[ [I + G_o(z)] \left[ C M C^T + V \right] \left[ I + G_o(z^{-1}) \right]^T = \]

\[ V + G_w(z) W G_w^T(z^{-1}) \]
KF return difference equality (SISO)

Assume that both, \( w(k) \in \mathcal{R} \) and \( y(k), \nu(k) \in \mathcal{R} \)

\[
[1 + G_o(z)][1 + G_o(z^{-1})] = \gamma [1 + \frac{W}{V} G_w(z) G_w(z^{-1})]
\]

\[
\gamma = \frac{V}{V + C M C^T}
\]
KF root locus for SISO Systems

\[ V(z) \]

\[ W(z) \]

\[ C \Phi(z) B_w \]

\[ G_w(z) \]

\[ + \]

\[ Y(z) \]

\[ + \]

\[ \sim Y^0(z) \]

\[ \sim Y^0(z) \]

\[ C \Phi(z) L \]

\[ G_o(z) \]

\[ [1 + G_o(z)] = [1 + C \Phi(z) L] = \frac{\hat{C}(z)}{\hat{A}(z)} \]

\[ c.l. \ poles \]

\[ o.l. \ poles \]

\[ G_w(z) = C \Phi(z) B_w = \frac{\hat{B}_w(z)}{\hat{A}(z)} \]

\[ o.l. \ zeros \]

\[ o.l. \ poles \]
KF root locus for SISO Systems

\[
\frac{\hat{C}(z^{-1})\hat{C}(z)}{\hat{A}(z^{-1})\hat{A}(z)} = \gamma \left[ 1 + \rho \frac{\hat{B}_w(z^{-1})\hat{B}_w(z)}{\hat{A}(z^{-1})\hat{A}(z)} \right]
\]

\[
\rho = \frac{W}{V} \geq 0
\]

input noise intensity

measurement noise intensity

\[
\gamma = \frac{V}{V + CMCT} > 0, \quad \text{for} \quad V \in (0, \infty)
\]
Utilizing LQR-KF duality, 

\[
|(1 + G_o(e^{j\omega}))| \geq \sqrt{\frac{V}{V + CMCT^T}}
\]

Therefore, a lower bound to the phase margin of \(G_o(e^{j\omega})\) is:

\[
PM \geq 2 \sin^{-1}\left\{0.5 \sqrt{\frac{V}{V + CMCT^T}}\right\}
\]
KF Loop gain margins (SISO)

Estimator was designed for $\gamma = 1$

Estimator is guaranteed to remain asymptotically stable for

$$\frac{1}{1 + \sqrt{V/(V + CMC^T)}} < \gamma < \frac{1}{1 - \sqrt{V/V + CMC^T}}$$
Summary

• Stationary Kalman filters (KF):
  – KF algebraic Riccati equation
  – Convergence properties

• Kalman filter / LQR duality

• KF return difference equality
  – Reciprocal root locus
  – Guaranteed robustness margins