ME233 Advanced Control II
Lecture 10

Infinite-horizon LQR
PART I

(ME232 Class Notes pp. 135-137)

LTI Optimal regulators (review)

- State space description of a discrete time LTI

$$x(k+1) \;=\; A\,x(k) + B\,u(k) \qquad\qquad x(0) = x_o$$

- Find optimal control $u^0(k),\;\; k = 0,\, 1,\, 2\, \cdots$

- That drives the state to the origin

$$x \rightarrow 0$$

Finite Horizon LQ optimal regulator (review)

LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

We want to find the optimal control sequence:

$$U_0^o = \left(u^o(0),\, u^o(1),\, \ldots,\, u^o(N-1) \right)$$

which minimizes the cost functional:

$$J[x(0)] = x^T(N) Q_f\, x(N) + \sum_{k=0}^{N-1} \left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right\}$$

LQ Cost Functional (review)

$$J[x(0)] = x^T(N)Q_f\,x(N) + \sum_{k=0}^{N-1}\left\{\begin{bmatrix} x(k) \\ u(k)\end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R\end{bmatrix}\begin{bmatrix} x(k) \\ u(k)\end{bmatrix}\right\}$$

- N — total number of steps—“horizon”
- $x^T(N)Q_f\,x(N)$ — penalizes the final state deviation from the origin
- $\begin{bmatrix} x(k) \\ u(k)\end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R\end{bmatrix}\begin{bmatrix} x(k) \\ u(k)\end{bmatrix}$ — penalizes the transient state deviation from the origin and the control effort

$$Q_f \succeq 0 \qquad \begin{bmatrix} Q & S \\ S^T & R\end{bmatrix} \succeq 0 \qquad R \succ 0$$

symmetric

Finite-horizon LQR solution (review)

$$J_k^o[x(k)] = x(k)^T P(k) x(k)$$

$$u^o(k) = -K(\underline{k+1})x(k)$$

$$K(k) = [B^T P(k) B + R]^{-1}[B^T P(k) A + S^T]$$

Where $P(k)$ is computed **backwards in time** using the *discrete Riccati difference equation* :

$$P(N) = Q_f$$

$$\begin{aligned} P(k-1) = {} & A^T P(k) A + Q \\ & - [A^T P(k) B + S][B^T P(k) B + R]^{-1}[B^T P(k) A + S^T] \end{aligned}$$

Properties of Matrix $\boldsymbol{P(k)}$ (review)

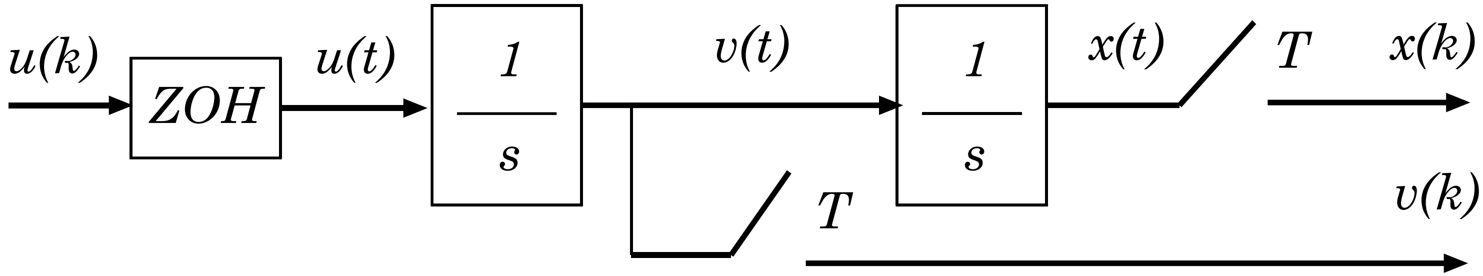
$P(k)$ satisfies:

1) $P(k) = P^T(k)$ (symmetric)

2) $P(k) \succeq 0$ (positive semi-definite)

Example – Double Integrator

Double integrator with ZOH and sampling time T =1:



$$x_1(k) \longleftrightarrow x(kT) \quad \textit{position}$$

$$x_2(k) \longleftrightarrow v(kT) \quad \textit{velocity}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

Example – Double Integrator

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

LQR cost:

$$J[x_o] = x^T(N) Q_f\, x(N) + \sum_{k=0}^{N-1} \underbrace{\left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right\}}_{x_1^T(k) x_1(k) + R u^2(k)}$$

Choose: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$R > 0$$

$$S = 0$$

$$P(N) = Q_f \succeq 0$$

only penalize position x_1 and control u

Example – Double Integrator (DI)

Compute ***P(k)*** for an arbitrary $P(N) = Q_f$ and *N*.

Computing backwards: $P(N) = Q_f$

$$P(k-1) = A^T P(k) A + Q - A^T P(k) B \left[B^T P(k) B + R \right]^{-1} B^T P(k) A$$

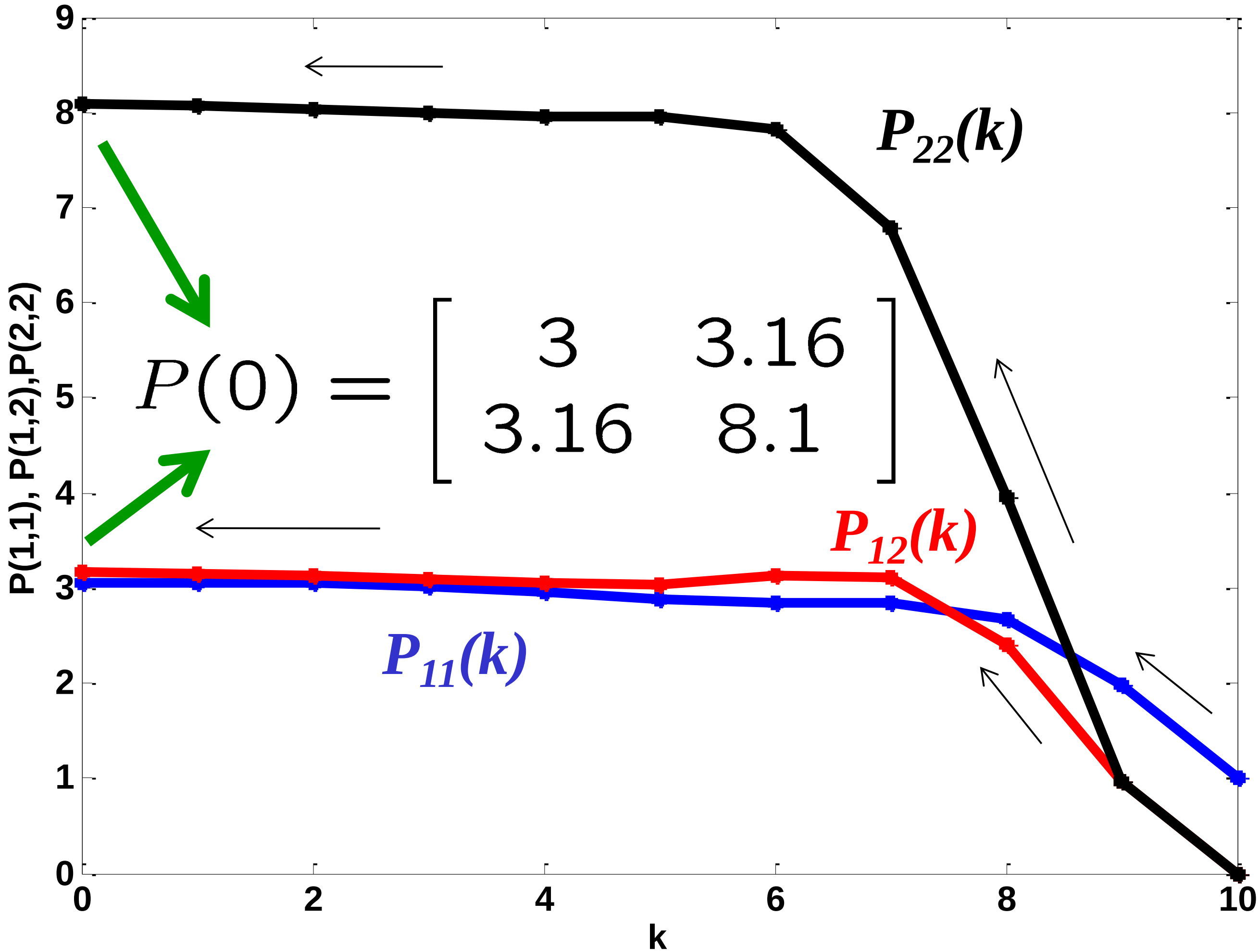
$$R > 0$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example – DI Finite Horizon Case 1

- $N = 10$, $R = 10$,

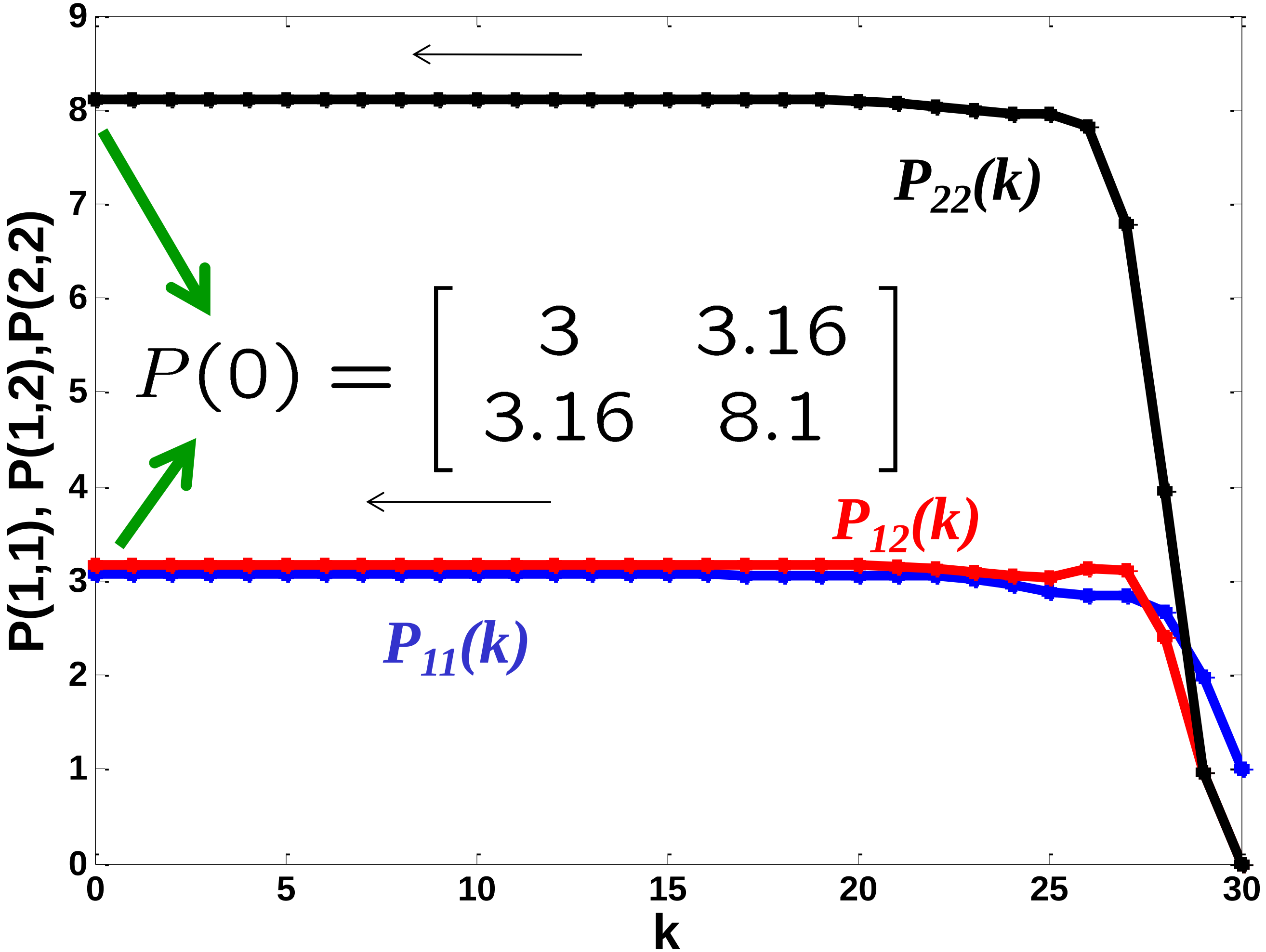
$$P(10) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



$$P(0) = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$$

Example – DI Finite Horizon Case 2

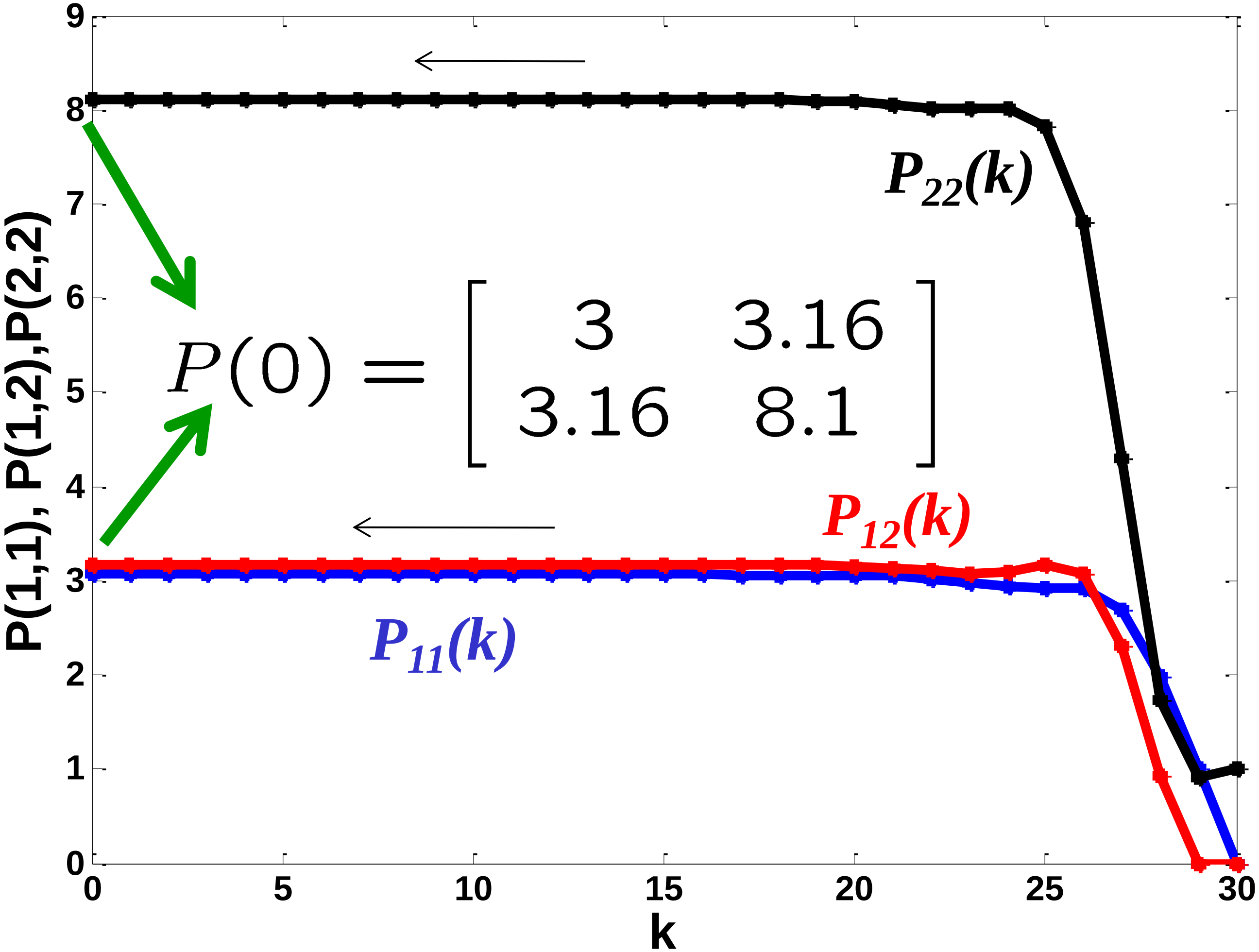
- $N = 30$, $R = 10$, $$P(30) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



$$P(0) = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$$

Example – DI Finite Horizon Case 3

- $N = 30$, $R = 10$, $$P(30) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



$$P(0) = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$$

Example – DI Finite Horizon

Observation:

In all cases, regardless of the choice of $P(N) = Q_f$

when the horizon, N, is sufficiently large

the backwards computation of the Riccati Eq. always converges to the same solution:

$$P(0) = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$$

Infinite-Horizon LQ regulator

LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

LQR that minimizes the cost:

$$J[x(0)] = x^T(N)Q_f\,x(N) + \sum_{k=0}^{N-1} \left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right\}$$

- We now consider the limiting behavior when

$$N \rightarrow \infty$$

Infinite Horizon (IH) LQ regulator

Consider the limiting behavior when $N \longrightarrow \infty$

LTI system:

$$x(k+1) \;=\; A\,x(k) + B\,u^o(k) \qquad x(0) = x_o$$

Optimal control:

$$u^o(k) = -K(k+1)\,x(k)$$
$$K(k) = [B^T P(k) B + R]^{-1} [B^T P(k) A + S^T]$$

Riccati equation:

$$P(N) = Q_f$$
$$P(k-1) = A^T P(k) A + Q - [A^T P(k) B + S][B^T P(k) B + R]^{-1}[B^T P(k) A + S^T]$$

Infinite Horizon LQ regulator question 1

Consider the limiting behavior when $N \longrightarrow \infty$

1) When does there exist a **<u>BOUNDED limiting</u>** solution

$$P(0) = P_\infty$$

to the Riccati Eq.

$$\begin{aligned} P(k-1) = {} & A^T P(k) A + Q \\ & - [A^T P(k) B + S][B^T P(k) B + R]^{-1}[B^T P(k) A + S^T] \end{aligned}$$

<u>for all</u> choices of $P(N) = Q_f = Q_f^T \succeq 0$?

Infinite Horizon LQ regulator question 2

Consider the limiting behavior when $N \longrightarrow \infty$

2) When does there exist a **UNIQUE limiting** solution

$$P(0) = P_\infty$$

to the Riccati Eq.

$$\begin{aligned} P(k-1) = {} & A^T P(k) A + Q \\ & - [A^T P(k) B + S][B^T P(k) B + R]^{-1}[B^T P(k) A + S^T] \end{aligned}$$

regardless of the choice of $P(N) = Q_f = Q_f^T \succeq 0$?

Infinite Horizon LQ regulator question 3

Consider the limiting behavior when $N \longrightarrow \infty$

3) When does the **limiting** solution

$$P(0) = P_\infty$$

to the Riccati Eq.

yield an **asymptotically stable** closed loop system?

$$A_c = A - BK_\infty$$

is Schur
(all eigenvalues inside unit circle)

$$K_\infty = \left[R + B^T P_\infty B\right]^{-1} \left[B^T P_\infty A + S^T\right]$$

LQ regulator Cost

$$J[x(0)] = x^T(N)Q_f\, x(N) + \sum_{k=0}^{N-1} \left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right\}$$

Define the square root of $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$, i.e.

Define the matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ such that

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$$

$$J[x(0)] = x^T(N)Q_f\, x(N) + \sum_{k=0}^{N-1} \left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right\}$$

LQ regulator Cost

$$J[x(0)] = x^T(N)Q_f\, x(N) + \sum_{k=0}^{N-1} \underline{\left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right\}}$$

- Define the fictitious output $\boldsymbol{p(k)}$ such that

$$p(k) = Cx(k) + Du(k)$$

$$J[x(0)] = x^T(N)Q_f\, x(N) + \sum_{k=0}^{N-1} \underline{\left\{ p^T(k)p(k) \right\}}$$

Infinite Horizon LQ optimal regulator

LTI system:

$$x(k+1) \;=\; A\,x(k) + B\,u(k) \qquad\qquad x(0) = x_o$$

$$p(k) = Cx(k) + Du(k)$$

Find optimal control which minimizes the cost functional:

$$J[x(0)] = x^T(N)Q_f\,x(N) + \sum_{k=0}^{N-1} \left\{ p^T(k)p(k) \right\}$$

Stabilizability Assumption

We are only interested in the case where the closed-loop dynamics are asymptotically stable

If *(A,B)* is not stabilizable, then there does not exist a control scheme that results is asymptotically stable closed-loop dynamics

⟹ For the infinite horizon optimal LQR problem, we always assume that *(A,B)* is stabilizable

Theorem 1 : Existence of a bounded $\boldsymbol{P}_{\infty}$

Let (A, B) be stabilizable

(uncontrollable modes are asymptotically stable)

Then, for $P(N) = Q_f = 0$, as $N \rightarrow \infty$

the "backwards" solution of the Riccati Eq.

$$
\begin{aligned}
P(k-1) = {} & A^T P(k) A + Q \\
& - [A^T P(k) B + S][B^T P(k) B + R]^{-1}[B^T P(k) A + S^T]
\end{aligned}
$$

converges to a **<u>BOUNDED limiting</u>** solution $P_{\infty} \succeq 0$

that satisfies the algebraic Riccati equation (DARE):

$$
\begin{aligned}
P_{\infty} = {} & A^T P_{\infty} A + Q \\
& - [A^T P_{\infty} B + S][B^T P_{\infty} B + R]^{-1}[B^T P_{\infty} A + S^T]
\end{aligned}
$$

Theorem 1 : Notes

- Theorem-1 only guarantees the existence of a bounded solution $P_\infty \succeq 0$ to the algebraic Riccati Equation

$$P_\infty = A^T P_\infty A + Q \\ - [A^T P_\infty B + S][B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

- The solution may not be unique, i.e. different final conditions $P(N) = Q_f$ may result in different limiting solutions $\boldsymbol{P}_\infty$ or may not even yield a limiting solution!

Theorem 2 : Existence and uniqueness of a positive definite asymptotic stabilizing solution

If *(A,B)* is stabilizable and the state-space realization $C(zI - A)^{-1}B + D$ has no transmission zeros, then

1) There exists a unique, bounded solution $P_\infty \succ 0$ to the DARE

$$P_\infty = A^T P_\infty A + Q \\ - [A^T P_\infty B + S][B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

2) The closed-loop plant $x(k+1) = [A - B\,K_\infty]\;x(k)$ is **asymptotically stable**

$$K_\infty = [B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

Theorem 3 : Existence of a stabilizing solution

If *(A,B)* is stabilizable and the state-space realization $C(zI - A)^{-1}B + D$ has no transmission zeros satisfying $|\lambda| \geq 1$, then

1) There exists a unique, bounded solution $P_\infty \succeq 0$ to the DARE

$$\begin{aligned} P_\infty = A^T P_\infty A + Q \\ - [A^T P_\infty B + S][B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T] \end{aligned}$$

2) The closed-loop plant $x(k+1) = [A - B\,K_\infty]\,x(k)$ is <u>**asymptotically stable**</u>

$$K_\infty = [B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

Theorem 4 : A different approach

The discrete algebraic Riccati equation (DARE) has a solution for which $A - BK_\infty$ is Schur <u>if and only if</u>

(A,B) is stabilizable and the state-space realization

$$G(z) = C(zI - A)^{-1}B + D$$

has no transmission zeros on the unit circle.

Moreover, $u^o(k) = -K_\infty x(k)$ is the optimal control policy that achieves asymptotic stability

$$\begin{aligned} P_\infty = {} & A^T P_\infty A + Q \\ & - [A^T P_\infty B + S][B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T] \end{aligned}$$

$$K_\infty = [B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

Special case: $S=0$

It turns out that the transmission zeros of

$$C(zI - A)^{-1}B + D$$

correspond to the unobservable modes of

$$(C, A)$$

(This will be assigned as a homework problem)

In Theorems 2 and 3, the transmission zeros condition becomes an observability/detectability condition

Theorem 2 : Existence and uniqueness of a positive definite asymptotic stabilizing solution, ***S* = 0**

If *(A,B)* is stabilizable and *(C,A)* is observable, then

1) There exists a unique, bounded solution $P_\infty \succ 0$ to the DARE

$$P_\infty = A^T P_\infty A + Q \\ - [A^T P_\infty B + S][B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

2) The closed-loop plant $x(k+1) = [A - B\,K_\infty]\;x(k)$ is **asymptotically stable**

$$K_\infty = [B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

Theorem 3 : Existence of a stabilizing solution, ***S* = 0**

If *(A,B)* is stabilizable and *(C,A)* is detectable, then

1) There exists a unique, bounded solution $P_\infty \succeq 0$ to the DARE

$$P_\infty = A^T P_\infty A + Q - [A^T P_\infty B + S][B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

2) The closed-loop plant $x(k+1) = [A - B\,K_\infty]\,x(k)$ is **asymptotically stable**

$$K_\infty = [B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

Theorem 4 : A different approach, **$S = 0$**

The discrete algebraic Riccati equation (DARE) has a solution for which $A - BK_\infty$ is Schur if and only if

(A,B) is stabilizable and *(C,A)* has no unobservable modes on the unit circle.

Moreover, $u^o(k) = -K_\infty x(k)$ is the optimal control policy that achieves asymptotic stability

$$
\begin{aligned}
P_\infty = {} & A^T P_\infty A + Q \\
& - [A^T P_\infty B + S][B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]
\end{aligned}
$$

$$K_\infty = [B^T P_\infty B + R]^{-1}[B^T P_\infty A + S^T]$$

Notes, ***S=0***

When ***(A,B)*** stabilizable and ***(C,A)*** observable or detectable, the infinite-horizon cost ($N \to \infty$) becomes

$$J[x_o] = \sum_{k=0}^{\infty} \left\{ x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

- The closed-loop plant is <u>asymptotically stable</u>

$$\Longrightarrow \quad \lim_{N \to \infty} x(N) = 0$$

- Solution of the DARE is unique, independent of ***P(N)***

Explanation: why is stabilizability needed

(A, B) not stabilizable $\Rightarrow$

there are unstable uncontrollable modes

$\Rightarrow$ there might be some initial conditions such that

$$\lim_{N\to\infty} J^o[x_o] = \infty$$

since the optimal cost is given by

$$J_N^o[x_o] = x_o^T P(0) x_0$$

$\Rightarrow$ $\lim_{N\to\infty} ||P(0)|| = \infty$

Explanation: why is detectability is needed, *S=0*

(C, A) not detectable ⟹

there are unstable unobservable modes

⟹ these modes do not affect the optimal cost

$$J[x_o] = \sum_{k=0}^{\infty} \left\{ x^T(k)\, Q\, x(k) + u^T(k)\, R\, u(k) \right\}$$

⟹ no need to stabilize these modes

Explanation: why is observability needed

The DARE can be written in the **Joseph stabilized** form:

$$A_c^T P_\infty A_c - P_\infty = -C^T C - K_\infty^T R K_\infty$$

$$A_c = [A - B K_\infty]$$ (closed-loop matrix)

$$Q = C^T C \succeq 0$$

$$R = D^T D \succ 0$$

define $\Longrightarrow$

$$\bar{C} = \begin{bmatrix} C \\ D K_\infty \end{bmatrix}$$

Explanation: why is observability needed

<u>Joseph stabilized</u> ARE:

$$A_c^T \, P_\infty \, A_c - P_\infty = -\bar{C}^T \, \bar{C}$$

Looks like a Lyapunov equation and, in fact, it is the Lyapunov equation for the **observability Grammian** of the pair

$$A_c = [A - B \, K_\infty] \qquad \bar{C} = \begin{bmatrix} C \\ D \, K_\infty \end{bmatrix}$$

Explanation: why is observability needed

Joseph stabilized ARE:

$$A_c^T P_\infty A_c - P_\infty = -\bar{C}^T \bar{C}$$

It can be shown that:

$[A\ C]$ observable $\Longleftrightarrow$ $[A_c\ \bar{C}]$ observable

$[A\ B]$ stabilizable $\Longleftrightarrow$ $P_\infty \succ 0$ and

$A_c = [A - B\,K_\infty]$ asymptotically stable (Schur)

Example – Double Integrator

LQR

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$J = \sum_{k=0}^{\infty} \left\{ y^2(k) + R\,u^2(k) \right\} \qquad R > 0$$

Example – Double Integrator

Penalize position in the infinite horizon cost functional:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

(C, A) Observable

(A, B) Controllable

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0.5 & 1.5 \\ 1 & 1 \end{bmatrix}$$

Example - Steady State Solution

- The steady state solution of the DARE:

$$A^T P A - P + C^T C - A^T P B \left[R + B^T P B \right]^{-1} B^T P A = 0$$

- Use matlab function dare

$$P = \texttt{dare}(A,\ B,\ C' * C,\ R)$$

- Get steady state answer: $P = \begin{bmatrix} 3 & 3.16 \\ 3.16 & 8.1 \end{bmatrix}$

Example - Infinite Horizon LQ Regulator

- The control law is given by:

$$u(k) = -K\,x(k) \qquad K = \left[R + B^T P B\right]^{-1} B^T P A$$

Answer ➔ $$K = \begin{bmatrix} 0.21 & 0.65 \end{bmatrix}$$

- Closed-loop poles are the eigenvalues of

$$A_c = A - B\,K$$

$$= \begin{bmatrix} 0.9 & 0.67 \\ -0.21 & 0.345 \end{bmatrix}$$

is Schur

- Use matlab command:

```
>> abs(eig(Ac))
ans =

    0.6736
    0.6736
```

Summary

- Convergence of LQR as horizon $N \to \infty$
  - (A, B) stabilizable
  - (C, A) detectable
- Infinite-horizon LQR
- Unique, positive definite solution of algebraic Riccati equation
- Closed-loop system is asymptotically stable

Additional Material (you are not responsible for this)

- Solutions of Infinite Horizon LQR using the Hamiltonian Matrix
  - (see ME232 class notes by M. Tomizuka)

- Strong and stabilizing solutions of the discrete time algebraic Riccati equation (DARE)

- Some additional results on the convergence of the asymptotic convergence of the discrete time Riccati equation (DRE)

Infinite Horizon LQ optimal regulator

Consider the nth order discrete time LTI system:

$$x(k+1) = A\,x(k) + B\,u(k) \qquad x(0) = x_o$$

We want to find the optimal control which minimizes the cost functional :

$$J = \sum_{k=0}^{\infty} \left\{ x^T(k)\, \underbrace{C^T C}_{Q}\, x(k) + u^T(k)\, R\, u(k) \right\}$$

Assume:

- (A, B) is controllable or asymptotically stabilizable
- (C, A) is observable or asymptotically detectable

Infinite Horizon LQR Solution:

$$J^o[x(0)] = x^T(0)\, P\, x(0)$$

$$u^o(k) = -K\, x(k)$$

$$K = \left[R + B^T P B\right]^{-1} B^T P A$$

Discrete time Algebraic Riccati (DARE) equation:

$$A^T P A - P + Q - A^T P B \left[R + B^T P B\right]^{-1} B^T P A = 0$$

Solution of the DARE

DARE:

$$A^T P A - P + Q - A^T P B \left[R + B^T P B\right]^{-1} B^T P A = 0$$

1) Assume that A is nonsingular and define the 2n x 2n ***Backwards*** Hamiltonian matrix:

$$H_b = \left[\begin{array}{c|c} A^{-1} & A^{-1} B\, R^{-1}\, B^T \\ -C^T C\, A^{-1} & A^T + C^T C\, A^{-1} B\, R^{-1}\, B^T \end{array} \right]$$

2) Compute its first $\boldsymbol{n}$ eigenvalues ($|\lambda_i| < 1$):

$$\{\lambda_1,\, \lambda_2,\, \cdots,\, \lambda_n \,|\, \lambda_{n+1},\, \cdots,\, \lambda_{2n}\}$$

Solution of the DARE

- The first n eigenvalues of $\boldsymbol{H}$ are the eigenvalues of

$$A_c = A - B\,K \quad \text{where} \quad K = \left[R + B^T P B\right]^{-1} B^T P A$$

and are all inside the unit circle, $|\lambda_i| < 1$
(I.e. asymptotically stable)

- The remaining eigenvalues of $\boldsymbol{H}$ satisfy:

$$\lambda_{n+i} = \frac{1}{\lambda_i} \qquad i = 1, \cdots, n$$

Solution of the DARE

3) For each ***unstable*** eigenvalue of $\boldsymbol{H}$ (***outside the unit circle***), compute its associated eigenvector :

$$H_b \underbrace{\begin{bmatrix} f_{n+i} \\ g_{n+i} \end{bmatrix}}_{v_{n+i}} = \lambda_{n+i} \underbrace{\begin{bmatrix} f_{n+i} \\ g_{n+i} \end{bmatrix}}_{v_{n+i}}$$

$$|\lambda_{n+i}| > 1$$

$$i = 1, \cdots, n$$

$$f_{n+i}, g_{n+i} \in \mathcal{C}^n$$

4) Define the $n \; x \; n$ matrices:

$$X_1 = \begin{bmatrix} f_{n+1} & f_{n+2} & \cdots & f_{2n} \end{bmatrix}$$

$$X_2 = \begin{bmatrix} g_{n+1} & g_{n+2} & \cdots & g_{2n} \end{bmatrix}$$

Solution of the ARE

5) Finally, $\boldsymbol{P}$ is computed as follows:

$$\boxed{P = X_2 X_1^{-1}}$$

- Matlab command **`dare`**: (Discrete ARE)

$$\boxed{[\,P,\,\Lambda,\,K\,,rr\,] = \mathtt{dare}(A,\,B,\,C^T C,\,R)}$$

$$P = X_2 X_1^{-1} \qquad \Lambda = \mathrm{Diag}(\lambda_1, \cdots, \lambda_n)$$

$$K = R^{-1} B^T P \qquad |\lambda_i| < 1$$

Strong Solution of the DARE

A solution $P = P^T \succeq 0$ of the DARE

$$A^T PA - P + Q - A^T PB \left[R + B^T PB \right]^{-1} B^T PA = 0$$

is said to be a **strong solution**

if the corresponding closed loop matrix $\boldsymbol{A_c}$

$$A_c = A - BK \qquad K = \left[R + B^T PB \right]^{-1} B^T PA$$

has all its eigenvalues on or inside the unit circle.

$$|\lambda_i(A_c)| \le 1; \; i = 1 \cdots n$$

Stabilizing Solution of the DARE

A strong solution $P = P^T \succeq 0$ of the DARE

$$A^T PA - P + Q - A^T PB \left[R + B^T PB\right]^{-1} B^T PA = 0$$

is said to be **stabilizing**

if the corresponding closed loop matrix $\boldsymbol{A_c}$

$$A_c = A - BK \qquad K = \left[R + B^T PB\right]^{-1} B^T PA$$

is Schur, i.e. it has all its eigenvalues inside the unit circle.

$$|\lambda_i(A_c)| < 1;\ i = 1 \cdots n$$

Theorem – Solutions to the DARE

Provided that (A,B) is stabilizable, then

i. the strong solution of the DARE exists and is unique.

ii. if (C,A) is detectable, the strong solution is the only nonnegative definite solution of the DARE.

iii. if (C,A) is has no unobservable modes on the unit circle, then the strong solution coincides with the stabilizing solution.

iv. if (C,A) has an unobservable mode on the unit circle, then there is no stabilizing solution.

Theorem – Solution to the DARE

Provided that ***(A,B)*** is stabilizable, then

v. if ***(C,A)*** has an unobservable mode inside or on the unit circle, then the strong solution is not positive definite.

vi. if ***(C,A)*** has an unobservable mode outside the unit circle, then as well as the the strong solution, there is at least one nonnegative definite solution of the DARE

S. W. Chan, G.C. Goodwin and K.S. Sin, "Convergence properties of the Riccati difference equation in optimal filtering of nonstabilizable systems, "*IEEE Trans. of Automatic Control* AC-29 (1984) pp 110-118.

Theorems - convergence of the DRE

Consider the "backwards" solution of the discrete time Riccati Equation

$$P(k-1) = C^T C + A^T P(k) A - A^T P(k) B \left[R + B^T P(k) B\right]^{-1} B^T P(k) A$$

$$P(N) = Q_f$$

1) Subject to

i. ***(A,B)*** is stabilizable and ***(C,A)*** is detectable,

ii. $Q_f \succeq 0$

then, as $N \rightarrow \infty$ ***P(k)*** converges exponentially to a unique **stabilizing** solution $\boldsymbol{P}_\infty$ of the DARE

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

Theorems - convergence of the DRE

Consider the "backwards" solution of the discrete time Riccati Equation

2) Subject to

i. ***(A,B)*** is stabilizable

ii. ***(C,A)*** is has no unobservable modes on the unit circle

iii. $Q_f \succ 0$

then, as $N \to \infty$ ***P(k)*** converges exponentially to a unique **stabilizing** solution $\boldsymbol{P}_\infty$ of the DARE

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

Theorems - convergence of the DRE

Consider the "backwards" solution of the discrete time Riccati Equation

3) Subject to

i. ***(A,B)*** is controllable

ii. $Q_f - P_\infty \succ 0$ or $Q_f = P_\infty$

then, as $N \rightarrow \infty$ ***P(k)*** converges to a unique **strong** solution $\boldsymbol{P}_\infty$ of the DARE

$$P_\infty = Q + A^T P_\infty A - A^T P_\infty B \left[R + B^T P_\infty B\right]^{-1} B^T P_\infty A$$

S. W. Chan, G.C. Goodwin and K.S. Sin, "Convergence properties of the Riccati difference equation in optimal filtering of nonstabilizable systems, "*IEEE Trans. of Automatic Control* AC-29 (1984) pp 110-118.