UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2016

Homework #3

Assigned: Mar. 2 (Wed)
Due: Mar. 8 (Tu)

1. In this problem we will verify some results concerning the convergence of the LQR's discrete Riccati equation (DRE) to a steady state solution and the existence, uniqueness and closed loop stability of the discrete algebraic Riccati equation (DARE) solution.

   Consider the design of an optimal LQR for the LTI discrete-time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned} \tag{1}$$

   where $u(k) = -K(k+1)\,x(k)$ is the optimal control input that minimizes the following cost criteria

$$J[x_o, m, Q_f, N] = x^T(N)\,Q_f\,x(N) + \sum_{k=m}^{N-1} \left\{ y^2(k) + R\,u^2(k) \right\} \quad \text{s.t.} \quad x(m) = x_o\,,$$

   for $m = 0$, $Q_f = Q_f^T \succeq 0$ and $R = R^T \succ 0$, and any arbitrary initial condition $x_o \in \mathcal{R}^n$. Define the optimal value function

$$J^o[x_o, m, Q_f, N] = \min_{U_{[m,N-1]}} J[x_o, m, Q_f, N]$$

   where $U_{[m,N-1]} = \{u(m), \cdots, u(N-1)\}$ is the set of all possible control actions from $k = m$ to $k = N-1$.

   (a) Let

$$A = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

   and verify that (A, B) is stabilizable but (C, A) is not detectable. Let $P(N) = Q_f$. For each of the four cases

   i. $Q_f = 0$
   ii. $Q_f = \text{diag}(0,\ 0,\ 1)$,
   iii. $Q_f = \text{diag}(1,\ 1,\ 1)$,
   iv. $Q_f = \text{diag}(10,\ 1,\ 1)$,

   do the following:

- For $x_o = [1 \;\; 0 \;\; 1]^T$, plot $J^o[x_o, N-m, Q_f, N]$ vs m. (Note that this will require computing $P(k)$, the solution of the Riccati difference equation, backwards from $P(50) = Q_f$.)
- Compute the solution of the DARE symbolically in terms of unknown elements of P_∞ and compare it with values of $P(0)$ and $P(1)$.

Discuss your results.

(b) Let

$$A = \begin{bmatrix} 0.8 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad R = 0.1\,.$$

Repeat part (a) for the two cases

i. $Q_f = 0$

ii. $Q_f = \text{diag}(0,\, 0,\, 1)$.

Discuss your results.

2. Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.08 & -1 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.34 \\ 0.3 \end{bmatrix} (u(k) + w(k))$$
$$y(k) = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k)$$

where $u(k)$ is a deterministic input and

- $E\{X(0)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $E\{X(0)X^T(0)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$
- $X(0)$ is Gaussian and $W(k)$ and $V(k)$ are white Gaussian sequences
- $m_w = E\{W(k)\} = 10, \quad E\{V(k)\} = 0$
- $E\left\{ \begin{bmatrix} W(k+j) - m_w \\ V(k+j) \end{bmatrix} \begin{bmatrix} (W(k) - m_w) & V(k) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(j) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \delta(j)$
- $E\left\{ \begin{bmatrix} W(k) - m_w \\ V(k) \end{bmatrix} X^T(0) \right\} = 0$

A Kalman filter is used to estimate the state of the system using the measurement sequence $y(k)$.

(a) Find the steady state values of the following time varying matrices and scalars:

- The a priori state estimation error covariance $M(k)$
- The a posteriori state estimation error covariance $Z(k)$
- The a priori output estimation error covariance

$$\Lambda_{\tilde{y}^o\tilde{y}^o}(k,0) = E\{|\tilde{y}^o(k)|^2\} = CM(k)C^T + V\,.$$

- The Kalman filter gains $L(k)$ and $F(k)$

You should find the steady state values of these quantities by recursively computing their values forwards in time until they converge to their respective steady state values.

(b) Plot the response of $\Lambda_{\tilde{y}^o \tilde{y}^o}(k, 0)$.

3. **Kalman filter with correlated input and measurement noise:**

Consider the discrete-time system given by

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{2}$$
$$y(k) = Cx(k) + v(k) \tag{3}$$

where $E\{x(0)\} = x_o$, $E\{w(k)\} = 0$, $E\{v(k)\} = 0$, $E\{(x(0) - x_o)(x(0) - x_o)^T\} = X_o$, $E\{(x(0) - x_o)w^T(k)\} = 0$, $E\{(x(0) - x_o)v^T(k)\} = 0$, and

$$E\left\{\begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W & S \\ S^T & V \end{bmatrix} \delta(k-j)$$

where $V \in \mathbb{R}^{m \times m}$ is positive definite matrix. The a-priori Kalman filter for this system can be written as

$$\hat{x}^o(k+1) = A\hat{x}^o(k) + Bu(k) + L(k)[y(k) - C\hat{x}^o(k)] \tag{4}$$
$$L(k) = [AM(k)C^T + S][CM(k)C^T + V]^{-1} \tag{5}$$
$$M(k+1) = AM(k)A^T + W - [AM(k)C^T + S][CM(k)C^T + V]^{-1}[CM(k)A^T + S^T] \tag{6}$$

with initial conditions $\hat{x}^o(0) = x_o$ and $M(0) = X_o$.

Derive Eqs. (4)–(6) using previously-derived results in Kalman filtering and noticing that Eqs. (2)–(3) can be written as

$$x(k+1) = A'x(k) + Bu(k) + w'(k) + SV^{-1}y(k)\,,$$

where $A' = A - SV^{-1}C$ and

$$E\left\{\begin{bmatrix} w'(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w'(j)^T & v(j)^T \end{bmatrix}\right\} = \begin{bmatrix} W' & 0 \\ 0 & V \end{bmatrix} \delta(k-j), \qquad W' = W - SV^{-1}S^T\,.$$