UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Mechanical Engineering
ME233 Advanced Control Systems II
Spring 2016

Homework #2

Assigned: Feb. 17 (Wed)
Due: Feb. 25 (Th)

1. A pair of random variables, X and Y have a joint probability density function (PDF)

$$p_{XY}(x,y) = \begin{cases} 1\ , & 0 \le y \le 2x \text{ and } 0 \le x \le 1 \\ 0\ , & \text{elsewhere} \end{cases}$$

   (a) Compute the marginal probability density functions

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y)dx, \text{ and } p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)dy$$

   (b) Compute the marginal mean $m_X = E\{X\} = \int_{-\infty}^{\infty} x p_X(x) dx$.

   (c) Compute the marginal variance of X.

$$\Lambda_{XX} = \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x) dx$$

   (d) Obtain an expression for the conditional probability density function $p_{X|Y}(x|y)$, i.e. the conditional PDF of X given the outcome $Y = y$ for $0 \le y \le 2$, where

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)}$$

   (e) Determine the conditional mean $E\{X|Y = y\}$, i.e. the expected value of X given the outcome $Y = y$ for $0 \le y \le 2$.

   (f) Determine the conditional mean $E\{X|Y = 0.5\}$.

   (g) Notice that the conditional mean $E\{X|Y\}$ can be thought of as a function of the random variable Y. Therefore, it is itself a random variable. Introducing the notation

$$m_{X|Y}(Y) = E\{X|Y\} = \int_{-\infty}^{\infty} x p_{X|Y}(x|Y) dx,$$

   prove that the expected value of the conditional mean $m_{X|Y}(Y)$ is equal to the marginal mean of X, i.e.

$$E\{m_{X|Y}(Y)\} = \int_{-\infty}^{\infty} m_{X|Y}(y) p_Y(y) dy = m_X = \int_{-\infty}^{\infty} x p_X(x) dx.$$

   Verify this result by computing $E\{m_{X|Y}(Y)\}$ and comparing it to m_X for the example above.

(h) Compute the variance of the conditional mean $m_{X|Y}(Y)$ for the example above.

$$\Lambda_{m_{X|Y}m_{X|Y}} = \int_{-\infty}^{\infty} (m_{X|Y}(y) - m_X)^2 p_Y(y) dy$$

You should find that $\Lambda_{m_{X|Y}m_{X|Y}} < \Lambda_{XX}$.

(i) Obtain an expression for the conditional variance of X given Y

$$\Lambda_{X|YX|Y}(Y) = E\{(X - m_{X|Y}(Y))^2|Y\} = \int_{-\infty}^{\infty} (x - m_{X|Y}(Y))^2 p_{X|Y}(x|Y) dx$$

for the example above. Notice that the conditional variance of X given Y, $\Lambda_{X|YX|Y}(Y)$ is also a random variable.

(j) Finally, compute the expected value of the conditional variance of X given Y,

$$E\{\Lambda_{X|YX|Y}(Y)\} = \int_{-\infty}^{\infty} \Lambda_{X|YX|Y}(y) p_Y(y) dy$$

for the example above and verify that

$$\Lambda_{XX} = \Lambda_{m_{X|Y}m_{X|Y}} + E\{\Lambda_{X|YX|Y}(Y)\}.$$

2. Consider the stochastic system

$$Y(k) - 0.5Y(k-1) = W(k) - 0.3W(k-1) \tag{1}$$

where $W(k)$ is a wide sense stationary (WSS) zero mean white random sequence with unit variance, i.e.

$$m_W = 0 \qquad \Lambda_{WW}(l) = E\{W(k+l)W(k)\} = \delta(l)$$

and $\delta(l)$ is the unit pulse function. In this problem, we will compare the theoretical value of the relevant covariances with empirical estimates of those quantities computed in a simulation. [1]

(a) Do a numerical simulation (in Matlab, Python, or Julia) of the response of this system for one sample sequence $w(k)$:

i. Generate the sample sequence $w(k)$ using `w = randn(N,1)` or equivalent, where `N` is a large number (e.g. 5000).

ii. Generate the sample output sequence $y(k)$ by propagating the system dynamics over time with a for loop.

iii. Generate and plot the estimates of the covariances and cross-covariances $\Lambda_{WW}(j)$, $\Lambda_{WY}(j)$, $\Lambda_{YW}(j)$, $\Lambda_{YY}(j)$, for $j = \{-10, -9, \cdots, 0, \cdots 10\}$.

[1] Since you will require an initial condition to perform time simulations, the output of the system given by Eq. (1) will not, strictly speaking be WSS. However, if the length of the sample sequence is taken to be sufficiently long, the relevant quantities will be approximately given by time averages.

(b) Determine the cross-covariance (cross-correlation) function

$$\Lambda_{YW}(l) = E\{Y(k+l)W(k)\}$$

and $\hat{\Lambda}_{YW}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YW}(l)$. Plot $\Lambda_{YW}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare the results with those empirically obtained from your simulation. Notice that $\Lambda_{YW}(l)$ is a causal sequence, i.e. $\Lambda_{YW}(l) = 0$ for $l < 0$ and all the poles of $\hat{\Lambda}_{YW}(z)$ will be inside the unit circle.

(c) Determine the cross-covariance (cross-correlation) function

$$\Lambda_{WY}(l) = E\{W(k+l)Y(k)\}$$

and $\hat{\Lambda}_{WY}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{WY}(l)$. Plot $\Lambda_{WY}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare the results with those empirically obtained from your simulation. Notice that $\Lambda_{WY}(l)$ is an anti-causal sequence, i.e. $\Lambda_{WY}(l) = 0$ for $l > 0$ and all the poles of $\hat{\Lambda}_{WY}(z)$ will be outside the unit circle.

(d) Determine the auto-covariance (auto-correlation) function

$$\Lambda_{YY}(l) = E\{Y(k+l)Y(k)\}$$

and $\hat{\Lambda}_{YY}(z) = \sum_{l=-\infty}^{\infty} z^{-l}\Lambda_{YY}(l)$. Plot $\Lambda_{YY}(l)$ for $l = \{-10, -9, \cdots, 0, \cdots 10\}$ and compare the results with those empirically obtained from your simulation. Notice that $\hat{\Lambda}_{YY}(z)$ will have poles both outside and inside the unit circle.

(e) Compute $\Lambda_{YW}(0)$ utilizing Eq. (1).

Hint: Multiply both sides of Eq. (1) by $W(k)$ and take expectations.

(f) Compute $\Lambda_{YW}(1)$ utilizing Eq. (1).

Hint: Multiply both sides of Eq. (1) by $W(k-1)$ and take expectations.

(g) Compute $\Lambda_{YY}(0)$ utilizing equation (1).

Hint: From Eq. (1) we have

$$Y(k) = 0.5Y(k-1) + W(k) - 0.3W(k-1)\ . \tag{2}$$

Square both sides of Eq. (2) and take expectations.

3. Let $X \sim N(10, 2)$, $V_1 \sim N(0, 1)$ and $V_2 \sim N(0, 2)$ be independent random variables. Assume that you are trying to make a measurement of X with two different instruments. Let $Y = X + V_1$ be the measurement of X using the first instrument and $Z = X + V_2$ be the measurement of X using the second instrument, where V_1 and V_2 are respectively the measurement noises of the first and second instruments.

   (a) Determine $m_{X|Y=9}$, i.e. the conditional expectation of X given that the first instrument yielded the measurement $Y = 9$.

   (b) Determine $m_{X|Z=11}$, i.e. the conditional expectation of X given that the second instrument yielded the measurement $Z = 11$.

(c) Determine $m_{X|(Y=9,Z=11)}$, i.e. the conditional expectation of X given that the first and second instruments respectively yielded the measurements $Y = 9$ and $Z = 11$.

4. A random variable X is repeatedly measured, but the measurements are noisy. Assume that the measurement process can be described by

$$Y(k) = X + V(k)$$

where $X, V(0), V(1), V(2), \ldots$ are jointly Gaussian random variables with

$$\begin{aligned} E\{X\} &= 0 & E\{X^2\} &= X_0 \\ E\{V(k)\} &= 0 & E\{V(k+j)V(k)\} &= \Sigma_V \delta(j) \\ E\{XV(k)\} &= 0\ . \end{aligned}$$

Let $y(k)$ be the k-th measurement (i.e. outcome of $Y(k)$) and let $\bar{y}(k) = \{y(0), \ldots, y(k)\}$.

(a) Obtain the least squares estimate of X given the $k+1$ measurements $y(0), \ldots, y(k)$ and the corresponding estimation error covariance, i.e. find $\hat{x}_{|\bar{y}(k)}$ and $\Lambda_{\tilde{X}_{|\bar{y}(k)}\tilde{X}_{|\bar{y}(k)}}$.
Hint: You do not need to invert a $(k+1) \times (k+1)$ matrix to find these quantities. Instead express

$$\Lambda_{\bar{y}(k)\bar{y}(k)} = A + uv^T$$

where A is a matrix that is easy to invert and u and v are vectors. In this case, the matrix inversion lemma says that

$$\Lambda^{-1}_{\bar{y}(k)\bar{y}(k)} = A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} u v^T A^{-1}\ .$$

(b) We now examine the case when $X_0 \to \infty$, i.e. when no prior information is available on X. Show the following:

$$\lim_{X_0 \to \infty} \left(\hat{x}_{|\bar{y}(k)} \right) = \frac{1}{k+1} \left[y(0) + y(1) + \cdots + y(k) \right]$$

$$\lim_{X_0 \to \infty} \left(\Lambda_{\tilde{X}_{|\bar{y}(k)}\tilde{X}_{|\bar{y}(k)}} \right) = \frac{\Sigma_V}{k+1}\ .$$