ME 233 Advanced Control II

Continuous-time results 4

Linear Quadratic Gaussian Loop Transfer Recovery

(ME233 Class Notes pp.LTR1-LTR9)
Outline

• Review of Feedback

• LQG stability margins

• LQG-LTR
Basic Feedback Transfer Functions (TF)

- $Y(s)$ is the controlled output
- $U(s)$ is the control input
- $E(s)$ is error signal fed to the controller
- $R(s)$ is the output reference
- $D(s)$ is the disturbance input
- $V(s)$ is the measurement noise

$$E_T(s) = R(s) - Y(s) \quad \text{“true” error signal}$$

$$E_T(s) = [I + G_o(s)]^{-1} [R(s) - D(s)] + [I + G_o(s)]^{-1} G_o(s) V(s)$$
Basic Feedback Transfer Functions (TF)

\[ E_T(s) = R(s) - Y(s) \quad \text{“true” error signal} \]

\[ E_T(s) = [I + G_o(s)]^{-1} [R(s) - D(s)] + [I + G_o(s)]^{-1}G_o(s)V(s) \]

\[ T(s) + S(s) = I \]
Basic Feedback Transfer Functions (TF)

\[ E_T(s) = R(s) - Y(s) \]

\[ T(s) + S(s) = I \]

\[ E_T(s) = [I + G_o(s)]^{-1} [R(s) - D(s)] + [I + G_o(s)]^{-1}G_o(s) V(s) \]

\[ S(s) \]

\[ T(s) \]

Frequency domain and singular values:

\[ \sigma_{\text{max}}[A(j\omega)] = (\lambda_{\text{max}}[A^*(j\omega)A(j\omega)])^{\frac{1}{2}} \]

\[ \sigma_{\text{min}}[A(j\omega)] = (\lambda_{\text{min}}[A^*(j\omega)A(j\omega)])^{\frac{1}{2}} \]

\[ Y(j\omega) = A(j\omega)U(j\omega) \]

\[ \|Y(j\omega)\| \geq \sigma_{\text{min}}[A(j\omega)] \|U(j\omega)\| \]

\[ \|Y(j\omega)\| \leq \sigma_{\text{max}}[A(j\omega)] \|U(j\omega)\| \]
Basic Feedback Transfer Functions (TF)

\[ E_T(s) = R(s) - Y(s) \quad T(s) + S(s) = I \]

\[ E_T(s) = [I + G_o(s)]^{-1} [R(s) - D(s)] + [I + G_o(s)]^{-1} G_o(s) V(s) \]

\[ S(s) \quad T(s) \]

Frequency domain:

1) \( \| R(j\omega) \| \) and \( \| D(j\omega) \| \) are normally large at low frequencies

\[ \sigma_{\text{max}}[S(j\omega)] < 1 \quad \text{at low frequencies} \]

2) \( \| V(j\omega) \| \) and \textbf{plant model uncertainties} are normally large at high frequencies

\[ \sigma_{\text{max}}[T(j\omega)] < 1 \quad \text{at high frequencies} \]
Basic Feedback Transfer Functions (TF)

\[ E_T(s) = [I + G_o(s)]^{-1} [R(s) - D(s)] + [I + G_o(s)]^{-1} G_o(s) V(s) \]

\[ S(s) \quad T(s) \]

\[ \sigma_{\max}[S(j\omega)] < 1 \quad \text{at low frequencies} \]

\[ \sigma_{\min}[G_o(j\omega)] >> 1 \]

\[ \sigma_{\max}[T(j\omega)] < 1 \quad \text{at high frequencies} \]

\[ \sigma_{\max}[G_o(j\omega)] << 1 \]
Bode’s integral theorem (SISO)  

$$S(s) = \frac{1}{1 + G_o(s)}$$

Let the open loop transfer function $G_o(s)$ have relative degree $\geq 2$ and let $p_1, p_2, \ldots, p_m$ be the unstable open loop poles (right hand plane)

$$\int_0^{\infty} \ln(|S(j\omega)|) \, dw = \pi \sum_{i=1}^{m} p_i$$

When $G_o(s)$ is stable,

$$\int_0^{\infty} \ln(|S(j\omega)|) \, dw = 0$$

Figure 3. Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.
Multivariable Nyquist Stability Criterion

roots of $A_c(s) = 0$ are the **closed loop** poles

roots of $A(s) = 0$ are the **open loop** poles

$L(s) = [I + G_o(s)]$

$$\text{det}[L(s)] = \frac{A_c(s)}{A(s)}$$

$N(0, \text{det}[L(s)], D) :$ number of counterclockwise encirclements around $0$

by $\text{det}[L(s)]$ when $s$ is along the Nyquist path $D$
Multivariable Nyquist Stability Criterion

\[ L(s) = [I + G_o(s)] \]

\[ N(0, \det[L(s)], D) = P - Z \]

\[ P = \# \text{ of unstable open loop poles} \]

\[ Z = \# \text{ of unstable closed loop poles} \]

\[ N(0, \det[L(s)], D) \]: number of counterclockwise encirclements around 0 by \( \det[L(s)] \) when \( s \) is along the Nyquist path \( D \)
Robust Stability

Nominal closed loop system
(asymptotically stable)

\[ L(s) = [I + G_o(s)] \]

Feedback system has robust stability
iff

\[ N(0, \det[L(s)], D) = N(0, \det[L'(s)], D) \]

when \( s \) is along the Nyquist path \( D \)

Actual system
\( \Delta(s) : \) output multiplicative uncertainty

\[ G'_o(s) = [I + \Delta(s)]G_o(s) \]
\[ L'(s) = [I + G'_o(s)] \]
Robust Stability

\[ G_0(s) = G(s)C(s) \]
\[ L(s) = [I + G_0(s)] \]

Robust stability iff

\[ N(0, \det[L(s)], D) = N(0, \det[L'(s)], D) \]

\[ 0 < \sigma_{\min} [I + [I + \epsilon \Delta(s)]G_0(s)] \quad \text{for all} \quad \epsilon \in [0, 1] \]

when \( s \) is along the Nyquist path \( D \)

\[ 0 < \sigma_{\min} [I + [I + \epsilon \Delta(j\omega)]G_0(j\omega)] \quad \text{for all} \quad \begin{cases} \epsilon \in [0, 1] \\ \omega \in [0, \infty) \end{cases} \]
Robust Stability

\[ L(s) = [I + G_0(s)] \]

\[ 0 < \sigma_{\min} [I + [I + \epsilon \Delta(j\omega)]G_0(j\omega)] \]

\[ 0 < \sigma_{\min} [I + G_0(j\omega) + \epsilon \Delta(j\omega)G_0(j\omega)] \]

\[ L(j\omega) \]

\[ 0 < \sigma_{\min} [I + \epsilon \Delta(j\omega)G_0(j\omega)[I + G_0(j\omega)]^{-1}] \]

\[ T(j\omega) \]

\[ 0 < 1 + \sigma_{\max} [\Delta(j\omega)T(j\omega)] \quad \Longleftrightarrow \quad \sigma_{\max} [T(j\omega)] < \frac{1}{\sigma_{\max} [\Delta(j\omega)]} \]
Robust Stability

\[
\sigma_{\text{max}} [T(j\omega)] < \frac{1}{\sigma_{\text{max}} [\Delta(j\omega)]}
\]

\[
T(s) = G_o(s)[I + G_o(s)]^{-1}
\]

at high frequencies when \( \sigma_{\text{max}} [G_o(j\omega)] << 1 \)

\[
\sigma_{\text{max}} [T(j\omega)] \approx \sigma_{\text{max}} [G_o(j\omega)] < \frac{1}{\sigma_{\text{max}} [\Delta(j\omega)]}
\]
Stationary LQR

Cost:

\[ J_s = \frac{1}{2} E \{ x^T(t) C_Q^T C_Q x(t) + u^T(t) R u(t) \} \]

- **Optimal control:**
  \[ u^o(t) = -K x(t) + r \]

Where the gain is obtained from the solution of the steady state LQR

\[ K = R^{-1} B^T P \]

\[ A^T P + PA + C_Q^T C_Q - PB R^{-1} B^T P = 0 \]
LQR robustness properties

Phase Margin $\geq 60^\circ$

Close loop system is stable for: \[ \frac{1}{2} < \gamma < \infty \]
Stationary Kalman Filter

- Kalman Filter Estimator:

\[
\frac{d}{dt} \hat{x}(t) = A \hat{x}(t) + B u(t) + L \tilde{y}(t)
\]

\[
\tilde{y}(t) = y(t) - C \hat{x}(t)
\]

\[
L = MC^T V^{-1}
\]

\[
AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM
\]
KF dual robustness properties

\[ Y(s) + \tilde{Y}(s) \xrightarrow{\gamma L} \Phi(s) \xrightarrow{\hat{X}(s)} C \xrightarrow{\hat{Y}(s)} \]

\[ \Phi(s) = (sI - A)^{-1} \]

\[ G_o(j\omega) = C\Phi(j\omega)L \]

\( \gamma = 1 \)

Phase Margin \( \geq 60^o \)

Close loop system is stable for:

\[ \frac{1}{2} < \gamma < \infty \]
"Fictitious" KF robustness properties

Phase Margin $\geq 60^\circ$

Close loop system is stable for:

$$\frac{1}{2} < \gamma < \infty$$
LQR example 1

Double integrator (example in pp ME232-143):

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u
\]

\[
J = \frac{1}{2} \int_0^\infty \left\{ x^T C_Q^T C_Q x + R u^2 \right\} \, dt
\]

with

\[
C_Q = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad R > 0
\]

Only position is penalized
LQR example 1

Double integrator (example in pp ME232-143):

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \quad R > 0
\]

\[
J = \frac{1}{2} \int_0^\infty \left\{ y^2 + R u^2 \right\} dt
\]

\[
G_Q(s) = C_Q(sI - A)^{-1}B = \frac{1}{s^2}
\]
LQR example 1 close loop poles

Double integrator (example in pp ME232-143):

\[
1 + \frac{1}{R} \frac{1}{s^4} = 0
\]

\[
R \to 0 \quad \Rightarrow \quad |p_{ci}|, \quad | - p_{ci}| \to \infty
\]
LQR example 1 margins

\[ r(s) \quad + \quad U(s) \quad B \quad \Phi(s) \quad X(s) \quad K \quad - \]

\[ G_0(s) = K \Phi(s) B \]

\[ G_0(s) = \frac{2.5(s + 1.26)}{s^2} \]

\[ GM = \infty \]

\[ PM = 65.5^\circ \]

Bode Diagram
Gm = Inf, Pm = 65.5 deg (at 2.76 rad/sec)

\[ R = 0.1 \]
LQR example $T(s)$

$$T(s) = \frac{G_o(s)}{1 + G_o(s)}$$

$$T(s) = \frac{2.5(s + 1.26)}{(s + 1.26)^2 + 1.26^2}$$

$R = 0.1$
Fictitious KF Feedback Loop example 1

Controller design parameters \( B_w, W, V \) are chosen

\[
W = 1 \quad V = R = 0.1 \quad B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

KF return difference equality = LQR return difference equality

\[
G_w(s) = G_Q(s)
\]
Fictitious KF example 1 margins

\[ G_0(s) = L \Phi(s) C \]

\[ G_0(s) = \frac{2.5(s + 1.26)}{s^2} \]

\[ V = 0.1 \]

Bode Diagram

\[ Gm = \infty, \ Pm = 65.5 \text{ deg (at 2.76 rad/sec)} \]

\[ GM = \infty \]

\[ PM = 65.5^\circ \]
LQR example 1 margins

\[ r(s) \rightarrow + E(s) \rightarrow B \rightarrow \Phi(s) \rightarrow X(s) \rightarrow K \]

\[ G_0(s) = K \Phi(s) B \]

\[ G_0(s) = \frac{2.5(s + 1.26)}{s^2} \]

\[ Bode Diagram \]

\[ G_m = \infty, \ P_m = 65.5 \text{ deg} \ (at 2.76 \text{ rad/sec}) \]

\[ GM = \infty \]

\[ PM = 65.5^\circ \]

\[ R = 0.1 \]
\[ \dot{x}(t) = A x(t) + B u(t) + B_w w(t) \]

\[ y(t) = C x(t) + v(t) \]
Stationary LQG

Cost:

\[ J_s = \frac{1}{2} E\{ x^T(t) C_Q^T C_Q x(t) + u^T(t) R u(t) \} \]

- Optimal control:

\[ u^o(t) = -K \hat{x}(t) \]

Where the gain is obtained from the solution of the steady state LQR

\[ K = R^{-1} B^T P \]

\[ A^T P + P A + C_Q^T C_Q - P B R^{-1} B^T P = 0 \]
Stationary LQG

- Kalman Filter Estimator:

\[
\frac{d}{dt} \hat{x}(t) = A \hat{x}(t) + B u(t) + L \tilde{y}(t)
\]

\[
\tilde{y}(t) = y(t) - C \hat{x}(t)
\]

\[
L = MC^T V^{-1}
\]

\[
AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM
\]
Stationary LQG Compensator

\[ U(s) = C_{LQG}(s) E(s) \]

\[ C_{LQG}(s) = K \left( sI - A + BK + LC \right)^{-1} L \]
LQG Loop Transfer

\[
C_{LQG}(s) = K \ (sI - A + BK + LC)^{-1} \ L
\]

\[
G(s) = C \ (sI - A)^{-1} \ B
\]
LQG Robustness Margins?

Unfortunately, there are no guaranteed robustness margins results for a general LQG controller.
Example -1 Double integrator

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

\[W = 1 \quad V = 0.1\]
LQG example 1

Double integrator (example in pp ME232-143):

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

\[
J = \frac{1}{2} \int_0^\infty \left\{ x^T C_Q^T C_Q x + R u^2 \right\} dt
\]

with

\[
C_Q = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad R = 0.1
\]
LQG example 1 margins

\[ R(s) \rightarrow C_{LQG}(s) \rightarrow G(s) \rightarrow Y(s) \]

\[ G_0(s) = G(s)C_{LQG}(s) \]

Margins could be much worse!

**Bode Diagram**

\[ G_m = 9.54 \text{ dB at } 3.08 \text{ rad/sec}, \quad P_m = 32.8 \text{ deg at } 1.37 \text{ rad/sec} \]

\[ GM_{db} = 9.5 \text{db} \]

\[ PM = 33^\circ \]
LQR example 1 margins

\[ G_0(s) = K \Phi(s) B \]

\[ G_0(s) = \frac{2.5(s + 1.26)}{s^2} \]

Bode Diagram

GM = \infty

PM = 65.5^\circ

\[ R = 0.1 \]
Fictitious KF Feedback Loop example 1

\[ W = 1 \]
\[ V = 0.1 \]

\[ r(s) + E(s) \rightarrow G_m = \text{Inf}, P_m = 65.5 \text{ deg (at 2.76 rad/sec)} \]

\[ Y(s) \rightarrow 10 \]
\[ -10 \]
\[ 0 \]
\[ 10 \]
\[ 100 \]
\[ 150 \]

Magnitude (dB)

\[ -180 \]
\[ -135 \]
\[ -90 \]

Phase (deg)

\[ G_o(s) = L\Phi(s)C \]

\[ G_o(s) = \frac{2.5(s + 1.26)}{s^2} \]

\[ GM = \infty \]

\[ PM = 65.5^0 \]
LQG – Loop Transfer Recovery

LQG-LTR was developed by Prof. John Doyle (when he was a M.S. student at MIT).


John Doyle

Other important contributions in Robust Control


• "Analysis of feedback systems with structured uncertainty ($\mu$),"

LQG – Loop Transfer Recovery

LQG-LTR is a robust control design methodology that uses the LQG control structure

- LQG-LTR is not an optimal control design methodology.

- LQG-LTR is not even a stochastic control design methodology.

- A fictitious Kalman Filter is used as a robust control design methodology.
  - Output noise intensity and input noise vector $(V & B_w)$ are used as design parameters – not true noise parameters.
Stationary LQG Compensator

\[ U(s) = C_{LQG}(s) E(s) \]

\[ C_{LQG}(s) = K \left( sI - A + BK + LC \right)^{-1} L \]
LQG-LTR Method 1

- How to make an LQG compensator *structure* robust to unmodeled *output* multiplicative uncertainties

\[ r(s) \xrightarrow{+} E(s) \xrightarrow{-} C_{LQG}(s) \xrightarrow{U(s)} G(s) \xrightarrow{} [I + \Delta(s)] \xrightarrow{} Y(s) \]

- \( \Delta(s) \) is a multiplicative uncertainty which is stable and bounded, i.e.

\[ \sigma_{\text{max}} [\Delta(j\omega)] \leq m(j\omega) < \infty \]
LQG-LTR Theorem 1

Let $G_o(s) = G(s)C_{LQG}(s)$ where

$$C_{LQG}(s) = K (sI - A + BK + LC)^{-1} L$$

And let $K$ be the state feedback gain that is obtained as follows

$$K = \frac{1}{\rho} N^{-1} B^T P_\rho$$
$$N = N^T \succ 0$$
$$R = \rho N$$

$$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$$

$\rho > 0$

*make LQR weight: $C_Q = C$*
Under the assumptions in the previous page

- If \( G(s) = C\Phi(s)B \) is square and has no unstable zeros, then point-wise in \( s \)

\[
\lim_{\rho \to 0} G(s) C_{LQG}(s) = C\Phi(s)L
\]
LQG-LTR Theorem 1

$K$ is the state feedback solution of the following LQR

$$J = \frac{1}{2} \int_{0}^{\infty} \left\{ x^T C C x + \rho \, u^T N u \right\} \, dt \quad N = N^T \succ 0$$

- $C$ is the state output matrix in:
  $$y(t) = C \, x(t) + v(t)$$

- $\rho > 0$ which is made very small, i.e.
  $$\rho \to 0 \quad \text{"cheap" control} \quad \text{LQR}$$
LQG-LTR Method 1

\[
\rho \to 0 \quad \text{"cheap" control LQR} \quad : \quad C_Q = C
\]

\[
\begin{align*}
E(s) & \\
C_{LQG}(s) & \quad U(s) & \quad G(s) & \quad [I + \Delta(s)]
\end{align*}
\]
LQG-LTR-Method 1

\[ K = \frac{1}{\rho}N^{-1}B^T P_{\rho} \]

\[ N = N^T > 0 \]

\[ A^T P_{\rho} + P_{\rho} A + C^T C - \frac{1}{\rho} P_{\rho} B N^{-1} B^T P_{\rho} = 0 \]

\[ K = \frac{1}{\rho}N^{-1}B^T P_{\rho} \]

\[ N = N^T > 0 \]

Make it approximate (point-wise in s)

\[ \rho \rightarrow 0 \]
Fictitious KF is the target system

Since the LTR procedure achieves:

\[ \lim_{\rho \to 0} G(s) C_{LQG}(s) = C \Phi(s) L \]

We need to determine the observer feedback \( L \) so that the target system has desirable properties

More on this later
Example -1 Double integrator

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

\[ G(s) = C \Phi(s) B = \frac{1}{s^2} \]

no unstable zeros
Design fictitious KF Target System

Design Parameters:

\[ B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ W = 1 \]

\[ V = 0.1 \]

\[ GM = \infty \]

\[ PM = 65.5^\circ \]
LTR procedure for computing $K$

1) For a small $\rho > 0$ compute:

$$K = \frac{1}{\rho} N^{-1} B^T P_\rho$$

where $P_\rho$ is the solution of

$$A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0$$

2) Check if $G(s) C_{LQG}(s) \approx C \Phi(s) L$ otherwise, decrease $\rho$ and repeat the process.
LQG-LTR-Method 1

\[ K = \frac{1}{\rho} N^{-1} B^T P_\rho \]
\[ N = N^T > 0 \]
\[ A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0 \]

Make it approximate (point-wise in s)

\[ \rho \to 0 \]
LQG-LTR KF example 1 \( G_o(s) = G(s) C_{LQG}(s) \)

\( C\Phi(j\omega)L \)

\( \rho = 1e^{-11} \)

\( \rho = 0.1 \)

\( \rho = 1e^{-6} \)

\( \rho = 1e^{-11} \)

\( \rho = 0.1 \)

\( \rho = 1e^{-6} \)
Fictitious KF design parameters

\[
\lim_{\rho \to 0} G(s) C_{LQG}(s) = C \Phi(s) L
\]

Select \( B_w, W, \) and \( V \) as design parameters to shape the open loop transfer function

\[
G_{okf}(s) = C \Phi(s) L
\]
Fictitious KF Feedback Loop

\[ G_{okf}(s) = C \Phi(s) L \]

Sensitivity and Complementary sensitivity Transfer Functions:

\[ S(s) = \left[ I + G_{okf}(s) \right]^{-1} \]

\[ r(s) \rightarrow U(s) \]

\[ T(s) = G_{okf}(s) \left[ I + G_{okf}(s) \right]^{-1} \]

\[ r(s) \rightarrow Y(s) \]
Simplify fictitious noise covariance description

KF gain $L$ is calculated by:

$$L = \frac{1}{\mu^2} MC^T$$

$$AM + MA^T = -BwB_w^T + \frac{1}{\mu^2} MCT^TCM$$

only the ratio $W/V$ is important

$\mu$: measurement noise standard deviation

$E\{w(t)w(t)^T\} = I\delta(t)$

$\Rightarrow W = I$

$E\{v(t)v(t)^T\} = \mu^2 I\delta(t)$

$\Rightarrow V = \mu^2 I$
Simplify fictitious noise covariance description

KF gain $L$ is calculated by:

$$L = \frac{1}{\mu^2} M C^T$$

$$AM + MAT^T = -B_wB_w^T + \frac{1}{\mu^2} M C^T C M$$

Return difference equality:

$$(1 + G_{okf}(s))(1 + G_{okf}(-s))^T = I + \frac{1}{\mu^2} G_w(s) G_w(-s)^T$$
Fictitious KF Feedback Loop Design

Design parameters:
- Fictitious input noise input vector:
- Fictitious output noise standard deviation: (affects bandwidth of close loop system)

Design equation: (return difference equation)

\[ \sigma_i [1 + G_{okf}(j\omega)] = \sqrt{1 + \left( \frac{\sigma_i[G_w(j\omega)]}{\mu} \right)^2} \]

\[ G_{okf}(s) = C\Phi(s)L \]
\[ G_w(s) = C\Phi(s)B_w \]

\( B_w \) affects zeros of \( G_w(s) \)

\( \mu \)

\( i^{th} \) singular value
Fictitious KF Feedback Loop Design

\[ G_{okf}(s) = C \Phi(s)L \]
\[ G_w(s) = C \Phi(s)B_w \]

\[ \sigma_i[1 + G_{okf}(j\omega)] = \sqrt{1 + \left( \frac{\sigma_i[G_w(j\omega)]}{\mu} \right)^2} \]

1. **Designer-specified shapes:** When
   (generally at low frequency)

   \[ \sigma_i[G_{okf}(j\omega)] \approx \frac{\sigma_i[G_w(j\omega)]}{\mu} \]

   use \( B_w \) to place zeros of \( G_w(j\omega) \)

   \[ \frac{\sigma_{min}[G_w(j\omega)]}{\mu} \gg 1 \]

   \[ \begin{align*}
   \sigma_i[T(j\omega)] & \approx 1 \\
   \sigma_i[S(j\omega)] & \approx \frac{1}{\sigma_i[G_{okf}(j\omega)]}
   \end{align*} \]
Fictitious KF Feedback Loop Design

\[ Y(s) = G_{okf}(s) X(s) \]
\[ G_{okf}(s) = C \Phi(s) L \]
\[ G_w(s) = C \Phi(s) B_w \]

\[ \sigma_i[1 + G_{okf}(j\omega)] = \sqrt{1 + \left( \frac{\sigma_i[G_w(j\omega)]}{\mu} \right)^2} \]

2. High frequency attenuation:

As \[ \omega \to \infty \]

\[ \sigma_i[G_{okf}(j\omega)] \approx \frac{\sigma_i[CL]}{\omega} \]

(gain Bode plot has -20 db/dec slope)
3. Well-behaved crossover frequency:

Sensitivity and complementary sensitivity TFs never become too large (even in the vicinity of the gain crossover frequency)

\[
\sigma_i[S(j\omega)] \leq 1 \quad \sigma_i[T(j\omega)] \leq 2 \quad \approx 6\text{ db}
\]
Fictitious KF Target Design

\[ r(s) + E(s) \xrightarrow{L} \Phi(s) \xrightarrow{X(s)} Y(s) \]

\[ W = I \]
\[ V = \mu^2 I \]

\[ L = \frac{1}{\mu^2} M C^T \]

\[ AM + M A^T = -B_w B_w^T + \frac{1}{\mu^2} M C^T C M \]

Goal: “Shape” the fictitious KF open loop transfer function

\[ G_{okf}(s) = C \Phi(s) L \]

\[ G_w(s) = C \Phi(s) B_w \]

- Design parameters:
  - \( B_w \) places zeros of \( G_w(s) \)
  - \( \mu \) adjusts gain crossover frequency of \( G_{okf}(s) \)
Example 1 – double integrator

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B u
\]

\[
y = C x
\]
Example 1: selection of $B_w$

$G_w(s) = C\Phi(s)B_w$

$B_w = \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix} = \begin{bmatrix} B_{w1} \\ 1 \end{bmatrix}$

$G_w(s) = \frac{B_{w1}s + 1}{s^2}$

sets the location of the zero
Example 1: selection of $B_w$

\[ G_w(s) = C \Phi(s) B_w \]

\[ G_w(s) = \frac{B_{w1}s + 1}{s^2} \]

In this example we will set $B_{w1} = 0$

\[ G_w(s) = \frac{1}{s^2} \]
Example 1: selection of $\mu$

\[ G_w(s) = \frac{1}{s^2} \]

$\mu$: adjusts gain crossover frequency of

\[ G_{okf}(s) = C \Phi(s) L \]

In this example we will set

$\mu = 0.01$
Example 1: selection of $\mu$

$$G_w(s) = \frac{1}{s^2}$$

$\mu = 0.01$

1. **Designer-specified shapes**: (low frequencies)

$$\sigma_{\min} \left[ \frac{G_w(j\omega)}{\mu} \right] \gg 1$$

$$\sigma_i[G_{okf}(j\omega)] \approx \frac{\sigma_i[G_w(j\omega)]}{\mu}$$
Example 1: selection of $\mu$

$$G_w(s) = \frac{1}{s^2}$$
$$\mu = 0.01$$

2. High frequency attenuation:

$$\omega \rightarrow \infty$$

$$\sigma_i[G_{okf}(j\omega)] \approx \frac{\sigma_i[CL]}{\omega}$$

(gain Bode plot has -20 db/dec slope)
Example 1: selection of $\mu$

\[ G_w(s) = \frac{1}{s^2} \]
\[ \mu = 0.01 \]

3. Well-behaved crossover frequency:

\[ \sigma_i[S(j\omega)] \leq 1 \]
\[ \sigma_i[T(j\omega)] \leq 2 \]
Example-2: Unstable Plant

\[ U \rightarrow B \rightarrow \Phi(s)^{-1} \rightarrow X \rightarrow C \rightarrow Y \]

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

\[ G(s) = C\Phi(s)B = \frac{1}{(s - 1)^2} \]

no unstable zeros
Example-2: I-action

- Introduce I-action to achieve 0 steady-state error to constant reference input

- Define I-action extended system
Example-2: I-action

- Define I-action extended system

\[ A_e = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_e = \begin{bmatrix} C & 0 \end{bmatrix} \]
Example-2: I-action

- I-action extended system

\[ W \rightarrow B_w \]

\[ U_e \rightarrow B_e \rightarrow \Phi_e(s) \rightarrow X \rightarrow C_e \rightarrow Y \]

\[ G_e(s) = (zI - A)^{-1} CB + U \]

\[
\begin{align*}
A_e &= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
B_e &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
C_e &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{align*}
\]
Example-2: selection of $B_w$

Design parameter:

\[
B_w = \begin{bmatrix} B_{w1} \\ B_{w2} \\ B_{w3} \end{bmatrix}
\]

We can "place" two zeros of $G_w(s)$

\[
G_w(s) = \frac{B_{w1}s + (B_{w2} - B_{w1})s + B_{w3}}{s(s - 1)^2}
\]

remember that, at low frequencies,

\[
\frac{\sigma_{\min}[G_w(j\omega)]}{\mu} >> 1 \quad \Rightarrow \quad \sigma_i[G_{okf}(j\omega)] \approx \frac{\sigma_i[G_w(j\omega)]}{\mu}
\]
Example-2: selection of $B_w$

\[
G_w(s) = C_e \Phi_e(s) B_w
\]

\[
G_w(s) = \frac{(s + 5)^2 + 5^2}{s(s - 1)^2}
\]

Example:
Place two zeros of $G_w(s)$
at \[z_{1,2} = -5 \pm 5j\]

\[
B_w = \begin{bmatrix} 1 \\ 11 \\ 50 \end{bmatrix}
\]
Example-2: selection of $\mu$

Example:

$\mu^2 = 0.01$

1. Designer-specified shapes: (low frequencies)

$$|G_{okf}(j\omega)| \approx \frac{|G_w(j\omega)|}{\mu}$$

for

$$\frac{|G_w(j\omega)|}{\mu} >> 1$$

$$G_w(s) = \frac{(s + 5)^2 + 5^2}{s(s - 1)^2}$$

$$W = I \quad V = \mu^2 I$$
Example-2: selection of $\mu$

$$G_{w}(s) = \frac{(s + 5)^2 + 5^2}{s(s - 1)^2}$$

$$W = I \quad V = \mu^2 I$$

**Example:** $\mu^2 = 0.01$

$$\frac{1}{\mu} \approx 32$$

$$G_{okf}(s) \approx \frac{44[(s + 4)^2 + 4.4^2]}{s(s - 1)^2}$$

2. High frequency attenuation:

$$\omega \rightarrow \infty \quad |G_{okf}(j\omega)| \approx \frac{CL}{\omega}$$

**Bode Diagram**

- $G_m = -16.8$ dB (at 7.3 rad/sec), $P_m = 76.4$ deg (at 43.5 rad/sec)
- $GM = -17$ dB
- $\omega_c = 44$ rad
- $PM = 76^O$
- High frequency attenuation: $-20$ dB/dec
Example-2: Fictitious KF Design

\[ G_w(s) = \frac{(s + 5)^2 + 5^2}{s(s - 1)^2} \]

\[ G_{okf}(s) \approx \frac{44[(s + 4)^2 + 4.4^2]}{s(s - 1)^2} \]

Example: \( \mu^2 = 0.01 \)

3. Well-behaved crossover frequency:

\[ |T(j\omega)| \leq 6\text{db} \]

\[ |S(j\omega)| \leq 0\text{db} \]
Example-2: selection of $B_w$

Close loop poles: As $\mu \to 0$

1. 2 close loop poles converge to the zeros of $G_w(s)$

$$G_w(s) = \frac{(s + 5)^2 + 5^2}{s(s - 1)^2}$$

2. The reminder pole goes to $-\infty$

Symmetric root locus:

$$W = I \quad V = \mu^2 I$$
Example-2: selection of $B_w$

\[
G_w(s) = C_e \Phi_e(s) B_w
\]

\[
W = I
\]

\[
V = \mu^2 I
\]

Return difference:

\[
1 + G_{ok_f}(s) = \frac{A_c(s)}{s(s-1)^2}
\]

Symmetric root locus:

We have the freedom to specify the location of the zero polynomial $B_w(s)$

\[
\frac{A_c(s)A_c(-s)}{s^2(s-1)^2(s+1)^2} = \left[ 1 + \frac{1}{\mu^2} \frac{B_w(s)B_w(-s)}{s^2(s-1)^2(s+1)^2} \right]
\]
Example-2: Fictitious KF Target Design

Open loop zeros

\[ z_{1,2} = -5 \pm 5j \]

\[
B_w = \begin{bmatrix}
1 \\
11 \\
50
\end{bmatrix}
\]

\[
\frac{A_c(s)A_c(-s)}{s^2(s - 1)^2(s + 1)^2} = \left[ 1 + \frac{1}{\mu^2} \frac{B_w(s)B_w(-s)}{s^2(s - 1)^2(s + 1)^2} \right]
\]
Example-2: LQG-LTR recovery

Use on extended system (including integrator dynamics)

\[ K = \frac{1}{\rho} N^{-1} B_e^T P_\rho \]

Keep decreasing \( \rho \) until

\[ G_e(s) C_{LQG}(s) \approx C \Phi(s) L \]

\[ A_e^T P_\rho + P_\rho A_e + C_e^T C_e - \frac{1}{\rho} P_\rho B_e N^{-1} B_e^T P_\rho = 0 \]
Example-2: LQG-LTR

Keep decreasing $\rho$ until

$$G_e(s) C_{LQG}(s) \approx C_e \Phi_e(s) L_e$$

$\rho = 10^{-5}$

$\rho = 10^{-11}$

$\rho = 10^{-17}$
Example-2: LQG-LTR

\[ R(s) \xrightarrow{C_{LQG}(s)} Y(s) \]

\[ G(s) \]

Keep decreasing \( \rho \) until

\[ G_e(s) C_{LQG}(s) \approx C_e \Phi_e(s) L_e \]

\[ \rho = 10^{-17} \]

Step Responses
- target
- LQG-LTR
LQG-LTR Method 2

- How to make an LQG compensator structure robust to unmodeled input multiplicative uncertainties

\[ r(s) \quad U(s) \quad [I + \Delta(s)] \quad G(s) \quad Y(s) \quad C_{LQG}(s) \]

- \( \Delta(s) \) is a multiplicative uncertainty which is stable and bounded, i.e.

\[ \sigma_{\text{max}} [\Delta(j\omega)] \leq m(j\omega) < \infty \]
LQG-LTR Theorem 2

Let \( G_o(s) = C_{LQG}(s) G(s) \) where

\[
C_{LQG}(s) = K \left(sI - A + BK + LC\right)^{-1} L
\]

And let \( L \) be the Kalman Filter feedback gain that is obtained as follows

\[
L = \frac{1}{\rho} M_{\rho} C^T N^{-1}
\]

\[
N = N^T > 0
\]

\[
AM_{\rho} + M_{\rho} A^T + BB^T - \frac{1}{\rho} M_{\rho} C^T N^{-1} C M_{\rho} = 0
\]

\[
\rho > 0
\]
Under the assumptions in the previous page

- If \( G(s) = C \Phi(s) B \) is square and has no unstable zeros, then point-wise in \( s \)

\[
\lim_{\rho \to 0} C_{LQG}(s) G(s) = K \Phi(s) B
\]
LQG-LTR Theorem 2

\( L \) is the Kalman Filter gain solution of the following filtering problem

\[
B_w = B \quad E\{w(k)w(k)^T\} = W = I
\]

\[
E\{v(k)v(k)^T\} = V = \rho N \succ 0
\]

- \( \rho > 0 \) which is made very small, i.e.

\[
\rho \rightarrow 0 \quad \text{“noiseless” output measurement}
\]
LQG-LTR Method 2

\[ \rho \to 0 \quad \text{“noiseless” output measurement} \quad B_w = B \]
More on LQG-LTR

- **LTR Theorem Proof**: Read ME233 Class Notes, pages LTR-3 to LTR-5 (also back of these notes)

- Fictitious Kalman Filter Design Techniques: Read ME233 Class Notes, pages LTR-6 to LTR-9

Outline

• Continuous time LQR stability margins

• Continuous time Kalman Filter stability margins

• Fictitious Kalman Filter

• LQG stability margins

• LQG-LTR
LQG-LTR Theorem 1

Assume that:

- \( G_0(s) = G(s) C_{LQG}(s) \) where
  \[
  C_{LQG}(s) = K (sI - A + BK + LC)^{-1} L
  \]
  - The feedback gain \( K \) is satisfies
  \[
  K = \frac{1}{\rho} N^{-1} B^T P_{\rho} \quad N = N^T \succ 0
  \]
  \[
  A^T P_{\rho} + P_{\rho} A + C^T C - \frac{1}{\rho} P_{\rho} B N^{-1} B^T P_{\rho} = 0
  \]

- If \( G(s) = C \Phi(s) B \) is square and has no unstable zeros, then point-wise in \( s \)
  \[
  \lim_{\rho \to 0} G(s) C_{LQG}(s) = C \Phi(s) L
  \]
Notation

• For convenience, we define:

\[ \Phi(s) = (sI - A)^{-1} \]

\[ \Phi_{LC}(s) = (sI - A + LC)^{-1} \]
Linear Algebra Result

• We often use results like:

\[ K \left[ I + \Phi(s)BK \right]^{-1} = \left[ I + K\Phi(s)B \right]^{-1} K \]

• which can be easily verified by multiplying left and right by the appropriate matrices:

\[ \left[ I + K\Phi(s)B \right] K = K \left[ I + \Phi(s)BK \right] \]

\[ K + K\Phi(s)BK = K + K\Phi(s)BK \]
Proof: The result is obtained in 4 steps:

**Step 1:** Alternate expression for the LQG compensator $C_{LQG}(s)$

\[ C_{LQG}(s) = \left[ I + K \Phi_{LC}(s) B \right]^{-1} K \Phi_{LC}(s) L \]

where

\[ \Phi_{LC}(s) = (sI - A + LC)^{-1} \]
Proof of Step 1

\[ C_{LQG}(s) = K (sI - A + BK + LC)^{-1} L \]

\[ = K \left( (sI - A + LC) + BK \right)^{-1} L \]

\[ = K \left[ I + \Phi_{LC}(s)BK \right]^{-1} \Phi_{LC}(s) L \]

\[ = \left[ I + K\Phi_{LC}(s)B \right]^{-1} K\Phi_{LC}(s) L \]
LQG-LTR – Theorem 1 Proof

**Step 2:** Let \( K(\rho) \) be given by

\[
K(\rho) = \frac{1}{\rho} N^{-1} B^T P_\rho
\]

where \( P_\rho \) is the solution of

\[
A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0
\]

( LTR procedure for computing \( K(\rho) \) )
LQG-LTR – Theorem 1 Proof

If \( G(s) = C\Phi(s)B \) has no unstable zeros

Then as \( \rho \to 0 \)

\[
K(\rho) \to \frac{1}{\sqrt{\rho}} N^{-1/2} T C
\]

where \( T \) is unitary, i.e.

\[
T^T T = I
\]
Lemma: maximally achievable accuracy of LQR

To proof step 2 we use the following lemma from:


Let \( P_\rho \) be the solution of the following algebraic Riccati equation

\[
A^T P_\rho + P_\rho A + C^T C - \frac{1}{\rho} P_\rho B N^{-1} B^T P_\rho = 0
\]

where \( N = N^T > 0 \) and \( G(s) = C \Phi(s) B \) is square.

Then

\[
G(s) = C \Phi(s) B \quad \text{has no unstable zeros if and only if} \quad \lim_{\rho \to 0} P_\rho = 0
\]
Sketch of proof of step 2

Rewriting the Riccati equation

\[ A^T P_\rho + P_\rho A + C^T C - \rho K^T(\rho)N K(\rho) = 0 \]

and utilizing \( P_\rho \rightarrow 0 \)

results in \( \rho K^T(\rho)N K(\rho) \rightarrow C^T C \)

Thus,

\[ K(\rho) \rightarrow \frac{1}{\sqrt{\rho}} N^{-1/2} T C \quad T^T T = I \]
LQG-LTR - Proof

**Step 3:** If \( G(s) = C \Phi(s) B \) is square and has no unstable zeros, then as \( \rho \to 0 \)

\[
C_{LQG}(s) \to [C \Phi_{LC}(s) B]^{-1} C \Phi_{LC}(s) L
\]

where

\[
\Phi_{LC}(s) = (sI - A + LC)^{-1}
\]
Proof of Step 3

\[ C_{\text{LQG}}(s) = [I + K \Phi_{\text{LC}}(s)B]^{-1} K \Phi_{\text{LC}}(s) L \]

substitute:

\[ K(\rho) \rightarrow \frac{1}{\sqrt{\rho}} N^{-1/2} T C \]

\[ C_{\text{LQG}}(s) \rightarrow \left[ \sqrt{\rho} T^T N^{1/2} + C \Phi_{\text{LC}}(s)B \right]^{-1} C \Phi_{\text{LC}}(s) L \]

\[ C_{\text{LQG}}(s) \rightarrow [C \Phi_{\text{LC}}(s)B]^{-1} C \Phi_{\text{LC}}(s) L \]
LQG-LTR – Theorem 1 Proof

Step 4: If \( G(s) = C \Phi(s) B \) is square and has no unstable zeros, then as \( \rho \to 0 \)

\[
C_{LQG}(s) \to [C \Phi(s) B]^{-1} [C \Phi(s) L]
\]

where

\[
\Phi(s) = (sI - A)^{-1}
\]
Proof of Step 4

\[ C_{LQG}(s) \rightarrow [C \Phi_{LC}(s)B]^{-1} C \Phi_{LC}(s) L \]

\[ C_{LQG}(s) \rightarrow \left[ C[sI - A + LC]^{-1} B \right]^{-1} C \Phi_{LC}(s) L \]

\[ C_{LQG}(s) \rightarrow \left[ C \left\{ \Phi(s)^{-1}[I + \Phi(s)LC] \right\}^{-1} B \right]^{-1} C \Phi_{LC}(s) L \]

\[ C_{LQG}(s) \rightarrow \left[ C[I + \Phi(s)LC]^{-1} \Phi(s)B \right]^{-1} C \Phi_{LC}(s) L \]

\[ C_{LQG}(s) \rightarrow \left[ [I + C \Phi(s)L]^{-1} C \Phi(s)B \right]^{-1} C \Phi_{LC}(s) L \]
Proof of Step 4

\[ C_{LQG}(s) \rightarrow \left[ [I + C\Phi(s)L]^{-1} C\Phi(s)B \right]^{-1} \Phi_{LC}(s) L \]

\[ C_{LQG}(s) \rightarrow [C\Phi(s)B]^{-1} \left[ I + C\Phi(s)L \right] \Phi_{LC}(s) L \]

\[ C_{LQG}(s) \rightarrow [C\Phi(s)B]^{-1} C \left[ I + \Phi(s)LC \right] \Phi_{LC}(s) L \]

\[ C_{LQG}(s) \rightarrow \left[ [C\Phi(s)B]^{-1} C\Phi(s) \left( sI - A + LC \right) \right] \Phi_{LC}(s) L \]

\[ \Phi_{LC}(s)^{-1} \]

\[ C_{LQG}(s) \rightarrow \left[ [C\Phi(s)B]^{-1} C\Phi(s) L \right] \]
LQG-LTR Theorem 2

Let:

- \( G_o(s) = C_{LQG}(s) G(s) \) where
  
  \[- C_{LQG}(s) = K \left( sI - A + BK + LC \right)^{-1} L \]

- The feedback gain \( L \) is satisfies

  \[ L = \frac{1}{\rho} M_\rho C^T N^{-1} \quad N = N^T > 0 \]

  \[ AM_\rho + M_\rho A^T + BB^T - \frac{1}{\rho} M_\rho C^T N^{-1} CM_\rho = 0 \]

- If \( G(s) = C\Phi(s)B \) is square and has no unstable zeros, then point-wise in \( s \)

  \[ \lim_{\rho \to 0} C_{LQG}(s) G(s) = K\Phi(s)B \]
Proof LQG-LTR Theorem 2

• Start with LQG-LTR Theorem 1

• Apply LQG – KF duality