ME 233 Advanced Control II

Continuous-time results 3

Linear Quadratic Gaussian (LQG) Optimal Control

(ME233 Class Notes pp.LQG1-LQG7)

Continuous time stationary LQG

Cost:

$$J_s = \frac{1}{2} E\{x^T(t) Q x(t) + u^T(t) R u(t)\}$$

- **Optimal control:** $u^o(t) = -K\,\hat{x}(t)$

Where the gain is obtained from the solution of the steady state LQR

$$K = R^{-1} B^T P$$

$$A^T P + P A + Q - P B R^{-1} B^T P = 0$$

Stationary LQG

Solution:

- **Kalman Filter Estimator:**

$$
\begin{aligned}
\frac{d}{dt}\widehat{x}(t) &= A\,\widehat{x}(t) + B\,u(t) + L\,\tilde{y}(t) \\
\tilde{y}(t) &= y(t) - C\,\widehat{x}(t)
\end{aligned}
$$

$$L = M\,C^T\,V^{-1}$$

$$AM + MA^T = -B_w W B_w^T + MC^T V^{-1} CM$$

Stationary LQG

Solution:

- **Optimal cost:**

$$J_s^o = \text{Tr}\left\{P\left[BKM + B_w W B_w^T\right]\right\}$$

Stationary LQG

Optimal cost (derivation)

The incremental optimal cost is

$$J_s^o = \lim_{T\to\infty} \frac{1}{T}\left\{\hat{J}^o + \int_0^T \mathrm{Tr}[QM(t)]dt + \mathrm{Tr}[SM(T)]\right\}$$

$$\begin{aligned}\hat{J}^o \;&=\; \frac{1}{2}x_o^T P(0)x_o + \frac{1}{2}\mathrm{trace}\,[P(0)X_o] \\ &+ \int_0^T \mathrm{trace}\{L^T(t)P(t)L(t)V(t)\}dt\end{aligned}$$

Thus

$$J_s^o = \mathrm{Tr}\left\{QM + L^T PLV\right\}$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \mathrm{Tr}\left\{QM + L^T PLV\right\}$$

Note:

$$Q = -A^T P - P A + P B R^{-1} B^T P$$

$$K = R^{-1} B^T P$$

$$L = M C^T V^{-1}$$

$$AM + M A^T = -B_w W B_w^T + M C^T V^{-1} C M$$

Stationary LQG

Optimal cost (derivation)

$$J_s^o = \text{Tr}\left\{QM + L^T PLV\right\}$$

last term:

$$\text{Tr}\left\{L^T PLV\right\} =$$

$$= \text{Tr}\{L^T PMC^T\} = \text{Tr}\{PMC^T L^T\}$$

$$= \text{Tr}\{PMC^T V^{-1} CM\}$$

$$= \text{Tr}\{P[AM + MA^T + B_w W B_w^T]\}$$

first term:

$$\text{Tr}\{QM\} = \text{Tr}\{\left[-A^T P - P A + P B K\right] M\}$$

$$= \text{Tr}\{-PMA^T - P AM + P B KM\}$$

Adding:

$$J_s^o = \text{Tr}\left\{P\left[BKM + B_w W B_w^T\right]\right\}$$